

The Language of Mathematics

A notation primer

This sheet explains the symbols and set phrases that every university mathematics course uses, usually without saying so. It goes with the video *The Language of Mathematics*. Keep it nearby: when a symbol in a lecture or in the book looks unfamiliar, look it up here and read it aloud the way the “quoted” text does. A one-page table of every symbol is at the end.

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1 Sets and elements

Video: Part 2

A *set* is a collection of objects, called its *elements*. A small set can be written by listing its elements between curly braces:

$$A = \{1, 2, 3\} \quad \text{“A is the set with elements one, two, three.”}$$

A set records only *which* objects belong to it. Order and repetition do not matter:

$$\{1, 2, 3\} = \{3, 1, 2\} = \{1, 1, 2, 3\}.$$

Symbol	Read aloud	Example
$x \in A$	“x is an element of A”, “x is in A”	$2 \in \{1, 2, 3\}$
$x \notin A$	“x is not an element of A”	$5 \notin \{1, 2, 3\}$
$\emptyset$ or $\{\}$	“the empty set”: the set with no elements	$\{x \in \mathbb{R} : x^2 = -1\} = \emptyset$

Capital letters (A, B, S) usually name sets and small letters (a, x) their elements, but this is a habit, not a rule.

The standard number sets

These sets are used so often that they have their own names, printed in *blackboard bold*.

Set	Name	Elements
$\mathbb{N}$	natural numbers	$1, 2, 3, \dots$ (the counting numbers)
$\mathbb{Z}$	integers	$\dots, -2, -1, 0, 1, 2, \dots$ (from German <i>Zahlen</i> , “numbers”)
$\mathbb{Q}$	rational numbers	fractions $\frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$ ($\mathbb{Q}$ for <i>quotient</i>)
$\mathbb{R}$	real numbers	every point of the number line, including $\sqrt{2}$ and π , which are not fractions
$\mathbb{C}$	complex numbers	numbers $a + bi$ with $a, b \in \mathbb{R}$ and $i^2 = -1$

Each set contains the one before it: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Watch out. Some books include 0 in $\mathbb{N}$. In Open Semester courses $\mathbb{N} = \{1, 2, 3, \dots\}$, and we write $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ when zero is wanted. Check the convention whenever you open a new book.

2 Describing sets by a rule: set-builder notation

Video: Part 3

Most interesting sets are infinite, so we describe them by a rule instead of a list. There are two forms.

Filter form. Say where the elements come from, then the condition they must pass:

$$\{x \in \mathbb{R} : x > 2\} \quad \text{“the set of all } x \text{ in } \mathbb{R} \text{ such that } x \text{ is greater than two.”}$$

The colon is read “*such that*”. Many books write a vertical bar instead, $\{x \in \mathbb{R} \mid x > 2\}$, with the same meaning. An object belongs to the set exactly when it comes from the stated place *and* passes the test: 3 is in this set, 2 is not (since $2 > 2$ is false), and neither is any number that is not real.

Build form. Give a formula and say what its parameter runs through:

$$\{2k : k \in \mathbb{Z}\} = \{\dots, -4, -2, 0, 2, 4, \dots\} \quad \text{“the set of all two } k \text{ such that } k \text{ is in } \mathbb{Z},”$$

which is the set of even integers.

Intervals

An interval is a piece of the number line. For real numbers $a < b$:

Notation	Meaning	Name
$[a, b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$	closed interval: both endpoints included
(a, b)	$\{x \in \mathbb{R} : a < x < b\}$	open interval: both endpoints left out
$[a, b), (a, b]$	one endpoint included, the other not	half-open intervals
$[a, \infty)$	$\{x \in \mathbb{R} : x \geq a\}$	∞ is not a number, so its side is always open

Watch out. (a, b) also denotes an ordered pair (Section 3). The context always tells which.

Two examples from Linear Algebra

Lecture 1 of Linear Algebra defines

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}$$

“ $\mathbb{R}^n$ is the set of all columns with entries x_1 to x_n , such that x_1 to x_n are real numbers.”

This is the build form: in words, $\mathbb{R}^n$ is the set of all ordered lists of n real numbers. Lecture 3 defines the span of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, again in build form:

$$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k : c_1, \dots, c_k \in \mathbb{R}\}.$$

3 Comparing and combining sets

Video: Part 4

Symbol	Read aloud	Meaning
$A \subseteq B$	“A is a subset of B”	every element of A is an element of B
$A \subsetneq B$	“A is a proper subset of B”	$A \subseteq B$ and $A \neq B$
$A = B$	“A equals B”	same elements: $A \subseteq B$ and $B \subseteq A$
$A \cup B$	“A union B”	$\{x : x \in A \text{ or } x \in B\}$
$A \cap B$	“A intersect B”	$\{x : x \in A \text{ and } x \in B\}$
$A \setminus B$	“A minus B”	$\{x : x \in A \text{ and } x \notin B\}$
$A \times B$	“A cross B”	$\{(a, b) : a \in A, b \in B\}$, all ordered pairs

Example: for $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, $A \cup B = \{1, 2, 3, 4\}$, $A \cap B = \{3\}$, $A \setminus B = \{1, 2\}$, and $\{1, 2\} \subseteq A$.

Watch out. Some books write $A \subset B$ for “subset” and others for “proper subset”. We always write $\subseteq$ or $\subsetneq$, which cannot be misread. In mathematics “or” is always *inclusive*: “ P or Q ” is true when P is true, when Q is true, and when both are.

Ordered pairs, n -tuples and $\mathbb{R}^n$

When order matters we use round brackets. An *ordered pair* (a, b) has a first entry a and a second entry b , and

$$(a, b) = (c, d) \iff a = c \text{ and } b = d.$$

So $(2, 3) \neq (3, 2)$, although the sets $\{2, 3\}$ and $\{3, 2\}$ are equal. An ordered list of n entries $(x_1, x_2, \dots, x_n)$ is an *n -tuple*, and two n -tuples are equal exactly when they agree in every position.

The set of all ordered pairs of real numbers is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the plane; the set of all n -tuples of real numbers is $\mathbb{R}^n$. Linear Algebra usually writes these lists as columns, which carry the same information.

Indices. The small numbers in $x_1, x_2, \dots, x_n$ are *subscripts* or *indices*: x_i “**x sub i**”, or simply “**x i**”, is the entry in position i . The phrase “for $i = 1, \dots, n$ ” means “for each of $i = 1, 2, \dots, n$ ”. The dots $\dots$ mean “and so on, in the same pattern”; they are used only when the pattern is obvious.

4 Logic: statements, quantifiers and implications

Video: Part 5

A *statement* is a sentence that is either true or false, such as “ $3 > 2$ ” (true) or “ $1 = 2$ ” (false). Mathematics is mostly written in words; these symbols make statements short and precise.

Symbol	Read aloud	Example
$\forall$	“for all”, “for every”	$\forall x \in \mathbb{R} : x^2 \geq 0$ (true)
$\exists$	“there exists”, “there is at least one”	$\exists x \in \mathbb{R} : x^2 = 2$ (true: $x = \sqrt{2}$)
$\exists!$	“there exists exactly one”	$\exists! x \in \mathbb{R} : x + 3 = 5$ ($x = 2$)
$P \Rightarrow Q$	“P implies Q”, “if P then Q”	$x = 2 \Rightarrow x^2 = 4$
$P \Leftrightarrow Q$	“P if and only if Q” (written “iff”)	$x^3 = 8 \Leftrightarrow x = 2$ (for $x \in \mathbb{R}$)
$\neg P, P \wedge Q, P \vee Q$	“not P”, “P and Q”, “P or Q”	mostly written in words

After $\forall x \in S$ or $\exists x \in S$ a colon or a comma is read “such that” or “we have”; some books write “s.t.”.

The order of quantifiers matters

$\forall x \in \mathbb{R} \exists y \in \mathbb{R} : x + y = 0$ is true: for each x , take $y = -x$.

$\exists y \in \mathbb{R} \forall x \in \mathbb{R} : x + y = 0$ is false: no single y works for every x .

In the first line y may depend on x ; in the second, y must be chosen first, once and for all. Read such lines slowly from left to right, one symbol at a time.

Negation. “Not every x has property P ” means “some x fails P ”, and “there is no x with property P ” means “every x fails P ”:

$$\neg(\forall x : P(x)) \iff \exists x : \neg P(x), \quad \neg(\exists x : P(x)) \iff \forall x : \neg P(x).$$

So a single *counterexample* disproves a “for all” statement: $x = \frac{1}{2}$ shows that “ $\forall x \in \mathbb{R} : x^2 \geq x$ ” is false, since $\frac{1}{4} < \frac{1}{2}$.

Implications

$P \Rightarrow Q$ promises only that *whenever* P is true, Q is true. Equivalent ways to say it: “if P , then Q ”, “ P is sufficient for Q ”, “ Q is necessary for P ”, “ P only if Q ”.

- The *converse* $Q \Rightarrow P$ is a different statement and may be false: $x = 2 \Rightarrow x^2 = 4$ is true, but $x^2 = 4 \Rightarrow x = 2$ is false (take $x = -2$).
- The *contrapositive* $\neg Q \Rightarrow \neg P$ says the same as $P \Rightarrow Q$: if $x^2 \neq 4$, then $x \neq 2$.
- $P \Leftrightarrow Q$ means both $P \Rightarrow Q$ and $Q \Rightarrow P$: the two statements are true in exactly the same cases. So a proof of an “if and only if” has two directions. Example from Linear Algebra: $\mathbf{x} = \mathbf{y} \iff x_i = y_i$ for every i .

Definitions and comparison signs

Symbol	Read aloud	Example
$:=$	“is defined to be” (the new name stands by the colon)	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$=, \neq$	“equals”, “is not equal to”	$(2, 3) \neq (3, 2)$
$<, \leq$	“is less than”, “is less than or equal to”	$2 \leq 2$ is true, $2 < 2$ is false
$>, \geq$	“is greater than”, “is greater than or equal to”	$x^2 \geq 0$
$\approx$	“is approximately equal to”	$\sqrt{2} \approx 1.414$

5 Functions

Video: Part 6

A function f from a set A to a set B assigns to each element of A exactly one element of B .

$$f : A \rightarrow B \quad \text{“}f \text{ is a function from } A \text{ to } B\text{.”}$$

A is the *domain* (the allowed inputs) and B the *codomain* (where the outputs live). The rule is written with the barred arrow $\mapsto$, read “maps to”:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto x^2 \quad \text{or equally} \quad f(x) = x^2. \quad \text{So } f(3) = 9.$$

Watch out. The plain arrow $\rightarrow$ goes between *sets* (from domain to codomain); the barred arrow $\mapsto$ goes between *elements* (from an input to its output).

The *image* (or *range*) of f is the set of outputs it actually takes, $f(A) = \{f(a) : a \in A\}$. For $x \mapsto x^2$ on $\mathbb{R}$ the codomain is $\mathbb{R}$ but the image is $[0, \infty)$. If $f : A \rightarrow B$ and $g : B \rightarrow C$, the *composition* $g \circ f : A \rightarrow C$, “g after f”, is $x \mapsto g(f(x))$: first f , then g .

Example from Linear Algebra: the length of a vector is a function $\mathbb{R}^n \rightarrow \mathbb{R}$, $\mathbf{v} \mapsto \|\mathbf{v}\|$.

6 Sums

Video: Part 6

Long sums are written with the capital Greek letter sigma, Σ , for “sum”:

$$\sum_{i=1}^n a_i := a_1 + a_2 + \cdots + a_n \quad \text{“the sum from } i \text{ equals one to } n \text{ of } a_i\text{.”}$$

The letter i is the *index*. It starts at the value below the sigma, goes up by one each step, and stops at the value above it. For example,

$$\sum_{i=1}^4 i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30.$$

The index is a placeholder: $\sum_{i=1}^n a_i$ and $\sum_{j=1}^n a_j$ are the same sum. Two rules follow directly from the definition (regroup the terms):

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i, \quad \sum_{i=1}^n c a_i = c \sum_{i=1}^n a_i.$$

In Linear Algebra, the dot product of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v_i w_i$, and a linear combination is $\sum_{i=1}^k c_i \mathbf{v}_i$. The capital pi, $\prod_{i=1}^n a_i := a_1 a_2 \cdots a_n$, does the same for products.

7 How mathematics is written

Video: Part 7

Mathematical texts are built from labelled blocks. The label tells you what kind of statement you are reading.

Label	What it is
Definition	Introduces a new term and fixes its exact meaning. Nothing to prove: it is an agreement about words.
Theorem	An important statement that has been proved.
Proposition	A proved statement, usually smaller or less central than a theorem.
Lemma	A helper statement, proved mainly to be used in the proof of something else.
Corollary	A statement that follows quickly from a theorem or proposition just before it.
Proof	The argument that shows a statement is true, using only definitions and results already established. Its end is marked ■, □ or “QED” (Latin <i>quod erat demonstrandum</i> , “which was to be shown”).
Example, Remark	Illustrations and comments; not part of the logical chain.

Here is a complete small piece of mathematics in this form.

Definition 7.1. An integer n is *even* if $n = 2k$ for some $k \in \mathbb{Z}$. Equivalently, the set of even integers is $\{2k : k \in \mathbb{Z}\}$.

Proposition 7.2. If m and n are even integers, then $m + n$ is even.

Proof. Let m and n be even. By the definition, $m = 2k$ and $n = 2j$ for some $k, j \in \mathbb{Z}$. Then

$$m + n = 2k + 2j = 2(k + j),$$

and $k + j \in \mathbb{Z}$. So $m + n$ is even, by the definition. □

Corollary 7.3. The sum of three even integers is even.

Proof. If a, b, c are even, then $a + b$ is even by Proposition 7.2, and so $(a + b) + c$ is even by the same proposition. □

Phrases you will meet in proofs

Phrase	Meaning
“Let $x \in A$.”	Take any element of A ; the argument must work whichever one it is.
“Suppose ...”	Assume this for the rest of the argument (or until a contradiction is reached).
“for some k ” / “for all k ”	there is at least one such k / every k works
“Hence”, “thus”, “therefore”, “so”	what follows is a consequence of what came before
“It suffices to show ...”	proving this smaller thing is enough to prove the claim
i.e. / e.g.	“that is” (restates) / “for example” (one case among others)
iff / s.t.	“if and only if” / “such that”
WLOG	“without loss of generality”: the other cases are the same up to renaming
“respectively”	“ a and b are 1 and 2, respectively” means $a = 1$ and $b = 2$: the lists match in order

Greek letters

Greek letters are used as extra names. The common ones:

α	alpha	β	beta	γ	gamma	δ, Δ	delta
ε	epsilon	θ	theta	λ	lambda	μ	mu
π	pi	σ, Σ	sigma	φ	phi	ω	omega

Typical uses: θ for angles, λ for eigenvalues (Linear Algebra, later), ε and δ for small positive numbers (Calculus).

8 Check yourself

Check yourself

1. Read aloud, then list the elements: $\{n \in \mathbb{Z} : -2 \leq n < 3\}$.
2. True or false: $\{1, 2, 2\} = \{2, 1\}$; $(1, 2) = (2, 1)$; $\frac{1}{2} \in \mathbb{Z}$; $\mathbb{Z} \subseteq \mathbb{Q}$.
3. For $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$, find $A \cup B$, $A \cap B$ and $A \setminus B$.
4. Which of these is true? (a) $\forall n \in \mathbb{N} \exists m \in \mathbb{N} : m > n$. (b) $\exists m \in \mathbb{N} \forall n \in \mathbb{N} : m > n$.
5. Compute $\sum_{i=1}^3 (2i - 1)$.
6. Give a counterexample to: “for all $x \in \mathbb{R}$, $x^2 > 0$.”

Answers. 1. “The set of all integers n such that n is at least minus two and less than three”: $\{-2, -1, 0, 1, 2\}$. 2. True; false; false; true. 3. $A \cup B = \{1, 2, 3, 4, 6\}$, $A \cap B = \{2, 4\}$, $A \setminus B = \{1, 3\}$. 4. (a) is true (take $m = n + 1$); (b) is false (no natural number is larger than all natural numbers, itself included). 5. $1 + 3 + 5 = 9$. 6. $x = 0$, since $0^2 = 0$ is not greater than 0.

9 Quick reference

Symbol	Read aloud	Example
$\{1, 2, 3\}$	the set with elements 1, 2, 3	order and repetition do not matter
$x \in A, x \notin A$	x is (is not) an element of A	$2 \in \{1, 2, 3\}, 5 \notin \{1, 2, 3\}$
$\emptyset$	the empty set	$\{x \in \mathbb{R} : x^2 < 0\} = \emptyset$
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	naturals, integers, rationals, reals, complex	$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
$\{x \in S : P(x)\}, \{x \in S \mid P(x)\}$	the set of x in S such that P(x)	$\{x \in \mathbb{R} : x > 2\}$
$\{f(t) : t \in T\}$	the set of all f(t) such that t is in T	$\{2k : k \in \mathbb{Z}\}$
$[a, b], (a, b)$	closed, open interval from a to b	$[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$
$A \subseteq B, A \subsetneq B$	A is a (proper) subset of B	$\{1, 2\} \subsetneq \{1, 2, 3\}$
$A \cup B, A \cap B$	A union B, A intersect B	$\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$
$A \setminus B$	A minus B	$\{1, 2, 3\} \setminus \{2\} = \{1, 3\}$
$A \times B$	A cross B: all ordered pairs	$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
$(a, b), (x_1, \dots, x_n)$	ordered pair, n-tuple	$(2, 3) \neq (3, 2)$
$\mathbb{R}^n$	$\mathbb{R}^n$: all n-tuples of real numbers	$(1, 0, -2) \in \mathbb{R}^3$
x_i	x sub i: the entry in position i	$x = (5, 7): x_2 = 7$
$\dots$	and so on, in the same pattern	$1, 2, \dots, n$
$\forall$	for all, for every	$\forall x \in \mathbb{R} : x^2 \geq 0$
$\exists, \exists!$	there exists (exactly one)	$\exists x \in \mathbb{R} : x^2 = 2$
: or	such that	$\{x : x > 0\}$
$\Rightarrow$	implies, if ... then	$x = 2 \Rightarrow x^2 = 4$
$\Leftrightarrow$, iff	if and only if	$x^3 = 8 \Leftrightarrow x = 2$
$\neg, \wedge, \vee$	not, and, or (inclusive)	$\neg(2 < 1)$
$:=$	is defined to be	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\leq, \geq, \neq, \approx$	at most, at least, not equal, approximately	$\pi \approx 3.14$
$f : A \rightarrow B$	f is a function from A to B	$f : \mathbb{R} \rightarrow \mathbb{R}$
$x \mapsto f(x)$	x maps to f(x)	$x \mapsto x^2$
$g \circ f$	g after f	$(g \circ f)(x) = g(f(x))$
$\sum_{i=1}^n a_i$	the sum from i = 1 to n of a i	$\sum_{i=1}^4 i = 10$
$\prod_{i=1}^n a_i$	the product from i = 1 to n of a i	$\prod_{i=1}^4 i = 24$
■, □, QED	end of proof	

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