

Problem Sheet 1: Solutions

Lectures 1 to 4. Part I covers Lectures 1 and 2, Part II covers Lectures 3 and 4, and Part III is practice on your own. Problems marked ► are solved in Recitation 1.

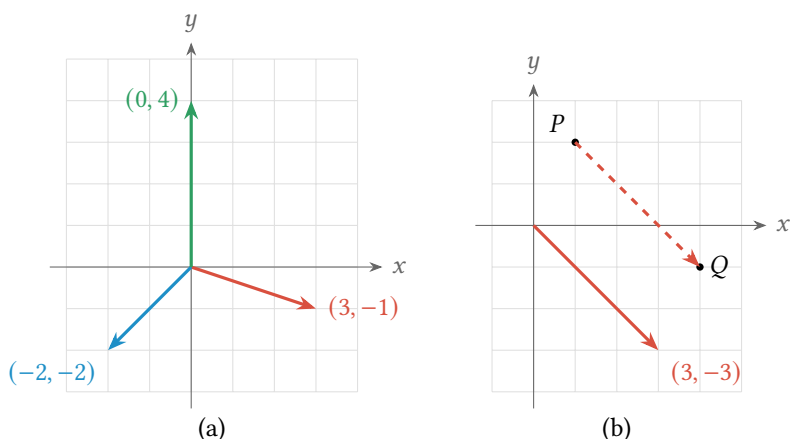
Part I. Vectors and their operations (Lectures 1 and 2)

A. Warm-up

Problem 1.

- (a) Draw, with tails at the origin, the vectors $(3, -1)$, $(-2, -2)$ and $(0, 4)$.
 (b) An arrow starts at $P = (1, 2)$ and ends at $Q = (4, -1)$. Which vector does it represent? Draw the same vector with its tail at the origin.

Solution. (a) The arrows end at the points $(3, -1)$, $(-2, -2)$ and $(0, 4)$ respectively; the third lies along the positive y -axis. (b) The arrow moves $4 - 1 = 3$ units across and $-1 - 2 = -3$ units up, so it represents $(3, -3)$. From the origin it ends at the point $(3, -3)$. In general the arrow from P to Q represents $Q - P$, computed component by component.



Problem 2. Let $\mathbf{u} = (1, -2, 4)$ and $\mathbf{v} = (3, 0, -1)$ in $\mathbb{R}^3$. Compute $\mathbf{u} + \mathbf{v}$, $3\mathbf{u}$, and $2\mathbf{u} - \mathbf{v}$.

Solution. $\mathbf{u} + \mathbf{v} = (1 + 3, -2 + 0, 4 - 1) = (4, -2, 3)$. $3\mathbf{u} = (3, -6, 12)$. $2\mathbf{u} - \mathbf{v} = (2, -4, 8) - (3, 0, -1) = (-1, -4, 9)$.

Problem 3. ► Recitation

Find all real numbers a and b such that $\begin{bmatrix} a+b \\ a-b \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Is there a pair (a, b) with $\begin{bmatrix} a+b \\ a-b \\ 2a \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$?

Solution. By the definition of equality the first equation means $a + b = 5$ and $a - b = 1$. Adding gives $2a = 6$, so $a = 3$, and then $b = 2$. These values satisfy both equations, so $(a, b) = (3, 2)$ is the unique solution. For the second equation the first two components force $(a, b) = (3, 2)$ as before, but then the third component is $2a = 6 \neq 4$. So no pair works: three equations in two unknowns can be inconsistent.

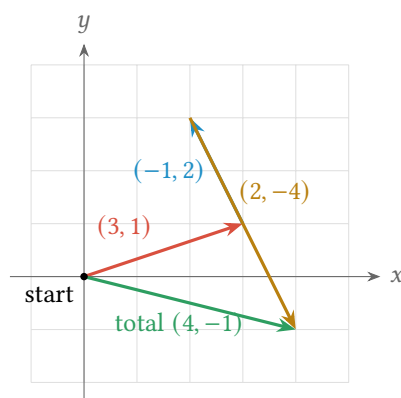
B. Core

Problem 4. ► Recitation

A hiker records three legs of a walk as displacement vectors (in km, east and north): $(3, 1)$, then $(-1, 2)$, then $(2, -4)$.

- What is the total displacement? Draw the three legs tip to tail and the total.
- Which single displacement brings the hiker straight back to the start?
- Does the total depend on the order in which the legs are walked? Explain using components.

Solution. (a) $(3, 1) + (-1, 2) + (2, -4) = (4, -1)$: four km east and one km south.



(b) $(-4, 1)$, the vector with $(4, -1) + (-4, 1) = \mathbf{0}$. (c) No. Each component of the total is a sum of real numbers, and real addition is commutative and associative, so the sums $3 - 1 + 2$ and $1 + 2 - 4$ do not depend on the order of their terms.

Problem 5. ► Recitation

Find numbers x and y such that $x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$. Then describe your answer in the row picture and in the column picture. (As in Lecture 1: the row picture intersects the lines given by the equations; the column picture combines the columns to reach the right-hand side.)

Solution. Comparing components, $2x + y = 7$ and $x + 3y = 11$. From the first, $y = 7 - 2x$; substituting, $x + 21 - 6x = 11$, so $x = 2$ and $y = 3$. Check: $2(2, 1) + 3(1, 3) = (4, 2) + (3, 9) = (7, 11)$. Row picture: the lines $2x + y = 7$ and $x + 3y = 11$ meet at the single point $(2, 3)$. Column picture: two copies of $(2, 1)$ followed by three copies of $(1, 3)$, tip to tail, reach the target $(7, 11)$.

Problem 6. ► Recitation

Colours on a screen are vectors (r, g, b) with each entry between 0 and 1. Let $\mathbf{c}_1 = (1, 0.4, 0)$ (orange) and $\mathbf{c}_2 = (0, 0.6, 1)$ (sky blue).

- Compute the even mix $\frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2$. What colour is it?
- For which numbers t between 0 and 1 is $t\mathbf{c}_1 + (1 - t)\mathbf{c}_2$ a shade of grey, that is, a vector with all three entries equal?

Solution. (a) $\frac{1}{2}(1, 0.4, 0) + \frac{1}{2}(0, 0.6, 1) = (0.5, 0.5, 0.5)$, a medium grey. (b) $t\mathbf{c}_1 + (1 - t)\mathbf{c}_2 = (t, 0.6 - 0.2t, 1 - t)$. Equal first and third entries need $t = 1 - t$, so $t = \frac{1}{2}$, and then the middle entry is $0.6 - 0.1 = 0.5$, which matches. So $t = \frac{1}{2}$ is the only such mix.

C. Proof and concept

Problem 7. ► Recitation

Using only the definitions of vector addition and equality in $\mathbb{R}^n$, prove that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Solution. Write $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$. By the definition of addition, the i -th component of $\mathbf{u} + \mathbf{v}$ is $u_i + v_i$, and the i -th component of $\mathbf{v} + \mathbf{u}$ is $v_i + u_i$. Since addition of real numbers is commutative, $u_i + v_i = v_i + u_i$ for every $i = 1, \dots, n$. By the definition of equality in $\mathbb{R}^n$, the two vectors are equal. $\square$

Problem 8. ► Recitation

Decide whether each statement is true for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Prove the true ones and give a counterexample for any false one.

- (a) If $2\mathbf{u} = 2\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.
- (b) If $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.
- (c) If $\mathbf{u}$ and $\mathbf{v}$ have the same first component, then $\mathbf{u} = \mathbf{v}$.

Solution. (a) True. Comparing i -th components, $2u_i = 2v_i$ for every i ; dividing by 2 gives $u_i = v_i$ for every i , so $\mathbf{u} = \mathbf{v}$. (b) True. Comparing components, $u_i + w_i = v_i + w_i$; subtracting w_i gives $u_i = v_i$ for every i . (c) False when $n \geq 2$: $\mathbf{u} = (1, 0, \dots, 0)$ and $\mathbf{v} = (1, 5, 0, \dots, 0)$ in $\mathbb{R}^n$ have the same first component but differ in the second. (For $n = 1$ the statement is true, since a vector in $\mathbb{R}^1$ has only one component.)

Part II. Linear combinations, span and the dot product (Lectures 3 and 4)

A. Warm-up

Problem 9. ► Recitation

Let $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 4)$ in $\mathbb{R}^3$.

- (a) Compute $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.
- (b) Find the cosine of the angle θ between $\mathbf{u}$ and $\mathbf{v}$. Is θ acute, right or obtuse?
- (c) Find the unit vector in the direction of $\mathbf{v}$.
- (d) For which number t is the vector $(1, t, 2)$ orthogonal to $\mathbf{u}$?

Solution. (a) $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 3 + 2 \cdot 0 + (-2) \cdot 4 = -5$, $\|\mathbf{u}\| = \sqrt{1 + 4 + 4} = 3$, $\|\mathbf{v}\| = \sqrt{9 + 0 + 16} = 5$.

(b) $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-5}{15} = -\frac{1}{3}$. The cosine is negative, so θ is obtuse ($\theta = \arccos(-\frac{1}{3}) \approx 109.5^\circ$).

(c) Divide $\mathbf{v}$ by its length: $\frac{1}{5}\mathbf{v} = (\frac{3}{5}, 0, \frac{4}{5})$. Check: $\frac{9}{25} + 0 + \frac{16}{25} = 1$.

(d) Orthogonal means the dot product is zero: $(1, t, 2) \cdot (1, 2, -2) = 1 + 2t - 4 = 2t - 3$, which is 0 exactly when $t = \frac{3}{2}$.

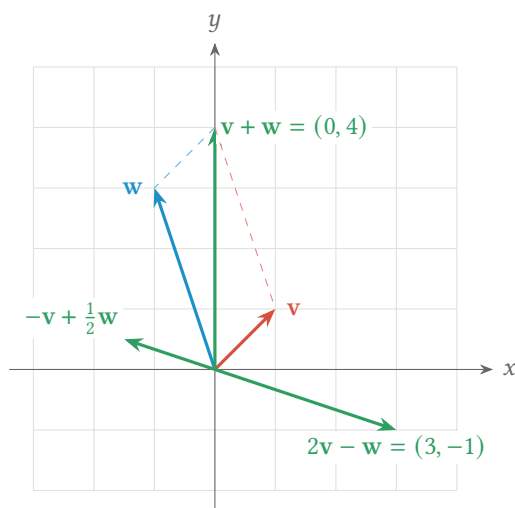
B. Core

Problem 10. ► Recitation

Let $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (-1, 3)$.

- On one grid, draw $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, $2\mathbf{v} - \mathbf{w}$ and $-\mathbf{v} + \frac{1}{2}\mathbf{w}$, all with tails at the origin.
- Find numbers x and y with $x\mathbf{v} + y\mathbf{w} = (-3, 5)$.
- Show that every vector $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^2$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. What is $\text{Span}\{\mathbf{v}, \mathbf{w}\}$?
- Would the answer to (c) change if $\mathbf{w}$ were replaced by $(-2, -2)$? Explain.

Solution. (a) The three combinations are $\mathbf{v} + \mathbf{w} = (0, 4)$, $2\mathbf{v} - \mathbf{w} = (2 + 1, 2 - 3) = (3, -1)$, $-\mathbf{v} + \frac{1}{2}\mathbf{w} = (-1 - \frac{1}{2}, -1 + \frac{3}{2}) = (-\frac{3}{2}, \frac{1}{2})$.



The dashed lines complete the parallelogram whose diagonal is $\mathbf{v} + \mathbf{w}$.

- Comparing components, $x - y = -3$ and $x + 3y = 5$. Subtracting the first equation from the second gives $4y = 8$, so $y = 2$ and $x = -1$. Check: $-(1, 1) + 2(-1, 3) = (-1 - 2, -1 + 6) = (-3, 5)$.
- We need $x - y = b_1$ and $x + 3y = b_2$. Subtracting gives $4y = b_2 - b_1$, so

$$y = \frac{b_2 - b_1}{4}, \quad x = b_1 + y = \frac{3b_1 + b_2}{4}.$$

These numbers exist for every b_1, b_2 , and substituting them back gives $\mathbf{b}$. So every vector of $\mathbb{R}^2$ is a combination of $\mathbf{v}$ and $\mathbf{w}$, and $\text{Span}\{\mathbf{v}, \mathbf{w}\} = \mathbb{R}^2$. (With $\mathbf{b} = (-3, 5)$ the formulas give $x = -1, y = 2$, as in (b).) This agrees with Theorem 1.3.8: $\mathbf{v}$ and $\mathbf{w}$ are not parallel, since $1 \cdot 3 - (-1) \cdot 1 = 4 \neq 0$.

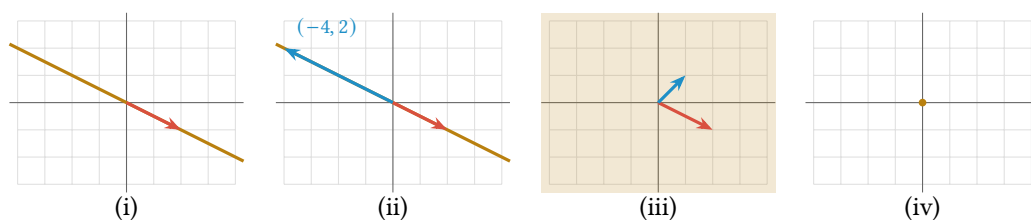
- Yes. $(-2, -2) = -2\mathbf{v}$, so every combination $x\mathbf{v} + y(-2, -2) = (x - 2y)\mathbf{v}$ is a multiple of $\mathbf{v}$. The span shrinks to the line $\text{Span}\{(1, 1)\}$ through the origin, and for example $(-3, 5)$ is no longer reachable. In the test of Theorem 1.3.8 the number is now $1 \cdot (-2) - (-2) \cdot 1 = 0$.

Problem 11. ► Recitation

- Sketch the span of each set in $\mathbb{R}^2$: (i) $\{(2, -1)\}$; (ii) $\{(2, -1), (-4, 2)\}$; (iii) $\{(2, -1), (1, 1)\}$; (iv) $\{(0, 0)\}$.

- (b) In $\mathbb{R}^3$ let $\mathbf{a} = (1, 0, 2)$, $\mathbf{b} = (0, 1, -1)$, $\mathbf{c} = (2, 3, 1)$ and $\mathbf{d} = (1, 1, 2)$. Decide whether $\mathbf{c}$ and $\mathbf{d}$ lie in $\text{Span}\{\mathbf{a}, \mathbf{b}\}$. Then describe $\text{Span}\{\mathbf{a}, \mathbf{b}\}$ by an equation, and explain why $\text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is the same set.

Solution. (a) (i) All multiples $t(2, -1)$: the line through the origin with equation $y = -\frac{1}{2}x$. (ii) Since $(-4, 2) = -2(2, -1)$, every combination is again a multiple of $(2, -1)$: the same line as in (i). (iii) The whole plane. For any (b_1, b_2) , solving $2s + t = b_1$, $-s + t = b_2$ gives $s = \frac{b_1 - b_2}{3}$ and $t = \frac{b_1 + 2b_2}{3}$. (iv) Only the origin: every multiple of the zero vector is the zero vector.



The span is shaded in gold: a line in (i) and (ii), the whole plane in (iii), a single point in (iv).

- (b) A combination of $\mathbf{a}$ and $\mathbf{b}$ is

$$s\mathbf{a} + t\mathbf{b} = (s, t, 2s - t).$$

For $\mathbf{c} = (2, 3, 1)$ the first two components force $s = 2$, $t = 3$, and then $2s - t = 1$ matches the third component. So $\mathbf{c} = 2\mathbf{a} + 3\mathbf{b}$ lies in the span. For $\mathbf{d} = (1, 1, 2)$ they force $s = t = 1$, but then $2s - t = 1 \neq 2$. So $\mathbf{d}$ is not in the span.

A vector (x, y, z) is of the form $(s, t, 2s - t)$ exactly when $z = 2x - y$ (take $s = x$, $t = y$). So $\text{Span}\{\mathbf{a}, \mathbf{b}\}$ is the plane $z = 2x - y$ through the origin.

Finally, since $\mathbf{c} = 2\mathbf{a} + 3\mathbf{b}$, any combination $p\mathbf{a} + q\mathbf{b} + r\mathbf{c} = (p + 2r)\mathbf{a} + (q + 3r)\mathbf{b}$ is already a combination of $\mathbf{a}$ and $\mathbf{b}$; and every combination of $\mathbf{a}, \mathbf{b}$ is one of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (take $r = 0$). So adding $\mathbf{c}$ does not enlarge the span: it is still the same plane. (This is Theorem 1.3.11: adding a vector that is already in the span does not change the span.)

Problem 12. ► Recitation

Let $A = (1, 0)$, $B = (4, 1)$ and $C = (0, 3)$.

- Draw the triangle ABC .
- Find the lengths of its three sides.
- Use dot products to find its three angles.
- Find the point D for which $ABDC$ is a square.

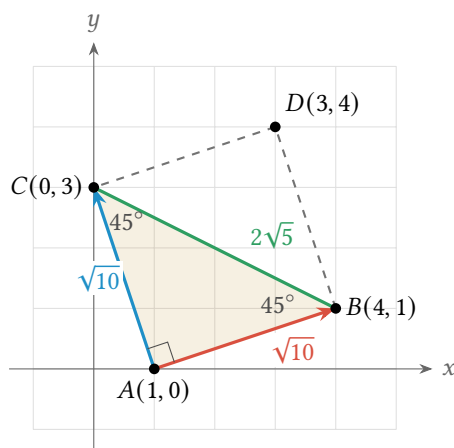
Solution. Work with the side vectors $\overrightarrow{AB} = B - A = (3, 1)$, $\overrightarrow{AC} = C - A = (-1, 3)$ and $\overrightarrow{BC} = C - B = (-4, 2)$.

(b) $|AB| = \sqrt{9 + 1} = \sqrt{10}$, $|AC| = \sqrt{1 + 9} = \sqrt{10}$, $|BC| = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$.

(c) At A : $\overrightarrow{AB} \cdot \overrightarrow{AC} = -3 + 3 = 0$, so the angle at A is 90° . At B : $\overrightarrow{BA} \cdot \overrightarrow{BC} = (-3, -1) \cdot (-4, 2) = 12 - 2 = 10$, so $\cos \angle B = \frac{10}{\sqrt{10}\sqrt{20}} = \frac{1}{\sqrt{2}}$ and $\angle B = 45^\circ$. At C : $\overrightarrow{CA} \cdot \overrightarrow{CB} = (1, -3) \cdot (4, -2) = 4 + 6 = 10$, so in the same way $\angle C = 45^\circ$. The triangle is a right isosceles triangle, and the angles add up to 180° . Pythagoras agrees: $10 + 10 = 20$.

(d) Going round the square $A \rightarrow B \rightarrow D \rightarrow C$, the side $\overrightarrow{BD}$ must equal $\overrightarrow{AC}$, so $D = B + \overrightarrow{AC} = (4, 1) + (-1, 3) = (3, 4)$. Check: $\overrightarrow{CD} = (3, 1) = \overrightarrow{AB}$, and all four sides have length $\sqrt{10}$ with a right angle at A .

(a) and (d) together:



C. Proof and concept

Problem 13. ► Recitation

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

- Prove the parallelogram law $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$, and say what it means for a parallelogram.
- Prove that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$. What does this say about the diagonals of a rhombus?
- True or false: if $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. Prove it or give a counterexample.

Solution. We use $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$ and the rules of the dot product from Lecture 4: it is symmetric ($\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$) and distributes over addition and subtraction.

(a) Expanding,

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v},$$

$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}.$$

Adding, the terms $\pm 2\mathbf{u} \cdot \mathbf{v}$ cancel and we get $2\mathbf{u} \cdot \mathbf{u} + 2\mathbf{v} \cdot \mathbf{v} = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$. □

In the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$, the diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$. So the sum of the squares of the two diagonals equals the sum of the squares of the four sides.

(b) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$. This is zero if and only if $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$, and since lengths are never negative, that happens if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$. □

A rhombus is a parallelogram with all sides equal, $\|\mathbf{u}\| = \|\mathbf{v}\|$. So its diagonals are perpendicular, and a parallelogram whose diagonals are perpendicular is a rhombus.

(c) False. Take $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$, $\mathbf{w} = (0, 2)$. Then $\mathbf{u} \cdot \mathbf{v} = 0 = \mathbf{u} \cdot \mathbf{w}$ but $\mathbf{v} \neq \mathbf{w}$. What is true is that $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$: the difference $\mathbf{v} - \mathbf{w}$ is orthogonal to $\mathbf{u}$, and need not be zero. (Compare Problem 8(b), where addition, unlike the dot product, can be cancelled.)

D. Challenge and application

Problem 14. ► Recitation

A search engine stores each document as a vector of word counts. With the four words (*vector*, *matrix*, *goal*, *team*), three short texts give

$$\mathbf{d}_1 = (4, 2, 0, 1), \quad \mathbf{d}_2 = (2, 2, 0, 0), \quad \mathbf{d}_3 = (0, 1, 3, 2).$$

The *cosine similarity* of two documents is the cosine of the angle between their vectors.

- (a) Compute the cosine similarity of each pair. Which two texts are most alike?
- (b) A fourth text is $\mathbf{d}_1$ copied out twice, so its vector is $2\mathbf{d}_1$. Show that its cosine similarity with any document $\mathbf{d}$ is the same as that of $\mathbf{d}_1$. Why is this a sensible property?
- (c) Explain why the cosine similarity of two word-count vectors (nonzero, with entries ≥ 0) always lies between 0 and 1, and when it equals 0.

Solution. (a) The lengths are $\|\mathbf{d}_1\| = \sqrt{21}$, $\|\mathbf{d}_2\| = \sqrt{8}$, $\|\mathbf{d}_3\| = \sqrt{14}$, and the dot products are $\mathbf{d}_1 \cdot \mathbf{d}_2 = 8 + 4 = 12$, $\mathbf{d}_1 \cdot \mathbf{d}_3 = 2 + 2 = 4$, $\mathbf{d}_2 \cdot \mathbf{d}_3 = 2$. So

$$\cos \theta_{12} = \frac{12}{\sqrt{21}\sqrt{8}} \approx 0.93, \quad \cos \theta_{13} = \frac{4}{\sqrt{21}\sqrt{14}} \approx 0.23, \quad \cos \theta_{23} = \frac{2}{\sqrt{8}\sqrt{14}} \approx 0.19.$$

Texts 1 and 2 are by far the most alike (both are mostly about vectors and matrices); text 3 is about sport.

(b) For any $c > 0$, $\|c\mathbf{d}_1\| = c\|\mathbf{d}_1\|$ and $(c\mathbf{d}_1) \cdot \mathbf{d} = c(\mathbf{d}_1 \cdot \mathbf{d})$, so

$$\frac{(c\mathbf{d}_1) \cdot \mathbf{d}}{\|c\mathbf{d}_1\| \|\mathbf{d}\|} = \frac{c(\mathbf{d}_1 \cdot \mathbf{d})}{c\|\mathbf{d}_1\| \|\mathbf{d}\|} = \frac{\mathbf{d}_1 \cdot \mathbf{d}}{\|\mathbf{d}_1\| \|\mathbf{d}\|}.$$

Take $c = 2$. Scaling does not change the direction of a vector, only its length, and the similarity depends only on direction. That is sensible: a longer text on the same topic should not count as a different topic.

(c) Every product $d_i e_i$ of two nonnegative entries is ≥ 0 , so $\mathbf{d} \cdot \mathbf{e} \geq 0$ and the cosine is ≥ 0 . By the Cauchy–Schwarz inequality, $\mathbf{d} \cdot \mathbf{e} \leq \|\mathbf{d}\| \|\mathbf{e}\|$, so the cosine is ≤ 1 . It equals 0 exactly when every product $d_i e_i$ is 0, that is, when the two texts have no word in common. The angle between two word-count vectors is therefore never more than 90° .

Problem 15. ► Recitation

- (a) Apply the Cauchy–Schwarz inequality to $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{1} = (1, \dots, 1)$ to prove

$$(x_1 + x_2 + \dots + x_n)^2 \leq n(x_1^2 + x_2^2 + \dots + x_n^2).$$

When does equality hold?

- (b) Deduce that if $a + b + c = 1$, then $a^2 + b^2 + c^2 \geq \frac{1}{3}$. For which a, b, c is it equal to $\frac{1}{3}$?
- (c) Use the triangle inequality to prove that $\|\mathbf{u} - \mathbf{w}\| \leq \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\|$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. What does it say about the distances between three points?

Solution. (a) Cauchy–Schwarz says $|\mathbf{x} \cdot \mathbf{1}| \leq \|\mathbf{x}\| \|\mathbf{1}\|$. Here $\mathbf{x} \cdot \mathbf{1} = x_1 + \dots + x_n$, $\|\mathbf{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$ and $\|\mathbf{1}\| = \sqrt{n}$. Squaring both sides (both are ≥ 0) gives the inequality. Equality in Cauchy–Schwarz

holds exactly when one vector is a multiple of the other. Since $\mathbf{1} \neq \mathbf{0}$, this means $\mathbf{x} = c \mathbf{1}$ for some c : all the x_i are equal. In words: the square of the average is at most the average of the squares.

(b) With $n = 3$, part (a) gives $1 = (a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, so $a^2 + b^2 + c^2 \geq \frac{1}{3}$. Equality needs $a = b = c$, and then $a + b + c = 1$ gives $a = b = c = \frac{1}{3}$.

(c) Write $\mathbf{u} - \mathbf{w} = (\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w})$ and apply the triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ with $\mathbf{x} = \mathbf{u} - \mathbf{v}$ and $\mathbf{y} = \mathbf{v} - \mathbf{w}$. $\square$

The distance between points $\mathbf{p}$ and $\mathbf{q}$ is $\|\mathbf{p} - \mathbf{q}\|$. So the inequality says that going straight from $\mathbf{u}$ to $\mathbf{w}$ is never longer than going via a third point $\mathbf{v}$: a detour never shortens the trip.

Part III. Practice on your own (Lectures 1 to 4)

These problems are not solved in the recitation. Try them on your own; full solutions are in the solutions booklet.

A. Warm-up

Problem 16. Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (1, 4, 0)$.

- (a) Compute $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u} - 2\mathbf{v}$.
- (b) Compute $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.
- (c) Is $\mathbf{u}$ orthogonal to $(1, 2, 0)$?

Solution. (a) $\mathbf{u} + \mathbf{v} = (2 + 1, -1 + 4, 3 + 0) = (3, 3, 3)$. $3\mathbf{u} = (6, -3, 9)$ and $2\mathbf{v} = (2, 8, 0)$, so $3\mathbf{u} - 2\mathbf{v} = (6 - 2, -3 - 8, 9 - 0) = (4, -11, 9)$.

(b) $\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + (-1) \cdot 4 + 3 \cdot 0 = 2 - 4 + 0 = -2$. $\|\mathbf{u}\| = \sqrt{4 + 1 + 9} = \sqrt{14}$ and $\|\mathbf{v}\| = \sqrt{1 + 16 + 0} = \sqrt{17}$.

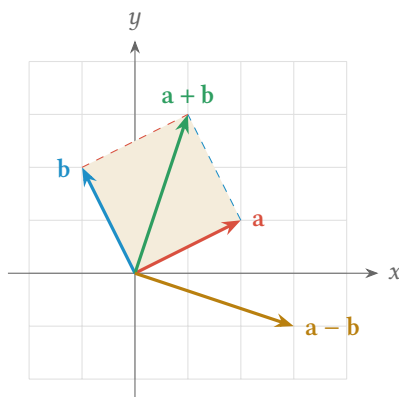
(c) Yes: $\mathbf{u} \cdot (1, 2, 0) = 2 \cdot 1 + (-1) \cdot 2 + 3 \cdot 0 = 2 - 2 + 0 = 0$.

B. Core

Problem 17. Let $\mathbf{a} = (2, 1)$ and $\mathbf{b} = (-1, 2)$ in $\mathbb{R}^2$.

- (a) Draw $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ on one grid.
- (b) Show that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and have the same length.
- (c) What shape do the points $\mathbf{0}$, $\mathbf{a}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{b}$ form? Find its area.
- (d) Find numbers x and y with $x\mathbf{a} + y\mathbf{b} = (0, 5)$.

Solution. (a) $\mathbf{a} + \mathbf{b} = (1, 3)$ and $\mathbf{a} - \mathbf{b} = (3, -1)$.



(b) $\mathbf{a} \cdot \mathbf{b} = 2 \cdot (-1) + 1 \cdot 2 = 0$, so they are orthogonal, and $\|\mathbf{a}\| = \sqrt{4+1} = \sqrt{5} = \sqrt{1+4} = \|\mathbf{b}\|$.

(c) A parallelogram whose sides $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and of equal length: a square with side $\sqrt{5}$ (shaded above). Its area is $(\sqrt{5})^2 = 5$.

(d) Comparing components, $2x - y = 0$ and $x + 2y = 5$. The first gives $y = 2x$; then $x + 4x = 5$, so $x = 1$ and $y = 2$. Check: $(2, 1) + 2(-1, 2) = (2 - 2, 1 + 4) = (0, 5)$.

Problem 18.

(a) Find all vectors in $\mathbb{R}^2$ that are orthogonal to $(2, -3)$, and describe this set geometrically.

(b) Which of these vectors have length $\sqrt{13}$?

(c) Show that there is no number k for which $(1, k, 2)$ and $(k, -1, 3)$ are orthogonal.

Solution. (a) $(x, y) \cdot (2, -3) = 2x - 3y = 0$ means $y = \frac{2}{3}x$, so the vectors are $(x, \frac{2}{3}x) = \frac{x}{3}(3, 2)$: all multiples $t(3, 2)$, $t \in \mathbb{R}$. Geometrically this is the line through the origin perpendicular to $(2, -3)$, which is the span of $(3, 2)$.

(b) $\|t(3, 2)\| = |t|\sqrt{9+4} = |t|\sqrt{13}$, which is $\sqrt{13}$ exactly when $t = \pm 1$. So the two vectors are $(3, 2)$ and $(-3, -2)$.

(c) $(1, k, 2) \cdot (k, -1, 3) = k - k + 6 = 6$ for every k . The dot product is never 0, so the two vectors are never orthogonal.

C. Proof and concept

Problem 19. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

(a) Prove: if $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{w}$ is orthogonal to every vector in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.

(b) True or false: if $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$, then $\mathbf{u} \cdot \mathbf{w} = 0$. Prove it or give a counterexample.

(c) Prove the Pythagorean theorem in $\mathbb{R}^n$: $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Solution. (a) A vector in the span has the form $s\mathbf{u} + t\mathbf{v}$. By the rules of the dot product, $\mathbf{w} \cdot (s\mathbf{u} + t\mathbf{v}) = s(\mathbf{w} \cdot \mathbf{u}) + t(\mathbf{w} \cdot \mathbf{v}) = s \cdot 0 + t \cdot 0 = 0$. □

(b) False. Take $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$ and $\mathbf{w} = (1, 0)$. Then $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$, but $\mathbf{u} \cdot \mathbf{w} = 1$. Orthogonality is not transitive.

(c) Expanding, $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$. This equals $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ exactly when $2\mathbf{u} \cdot \mathbf{v} = 0$, that is, when $\mathbf{u} \cdot \mathbf{v} = 0$. □

D. Challenge and application

Problem 20. A shop sells three items at prices $\mathbf{p} = (2, 5, 3)$ (in euros), and a customer buys the quantities $\mathbf{q} = (4, 1, 6)$.

- (a) Explain why the total cost is $\mathbf{p} \cdot \mathbf{q}$, and compute it.
- (b) All prices rise by 10%. Write the new price vector and the new total cost, and explain why the cost also rises by exactly 10%.
- (c) (Challenge) For positive numbers a, b, c , apply the Cauchy–Schwarz inequality to $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}})$ to prove

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9.$$

When does equality hold?

Solution. (a) Each item contributes its price times the quantity bought, and the total adds these up: exactly the dot product. $\mathbf{p} \cdot \mathbf{q} = 2 \cdot 4 + 5 \cdot 1 + 3 \cdot 6 = 8 + 5 + 18 = 31$ euros.

(b) The new price vector is $1.1 \mathbf{p} = (2.2, 5.5, 3.3)$, and the new cost is $(1.1 \mathbf{p}) \cdot \mathbf{q} = 1.1 (\mathbf{p} \cdot \mathbf{q}) = 1.1 \cdot 31 = 34.1$ euros. Check: $2.2 \cdot 4 + 5.5 \cdot 1 + 3.3 \cdot 6 = 8.8 + 5.5 + 19.8 = 34.1$. The cost rises by exactly 10% because the dot product is linear in each vector: scaling $\mathbf{p}$ by 1.1 scales $\mathbf{p} \cdot \mathbf{q}$ by 1.1.

(c) Let $\mathbf{x} = (\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $\mathbf{y} = (\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}})$. Then $\mathbf{x} \cdot \mathbf{y} = 1 + 1 + 1 = 3$, $\|\mathbf{x}\|^2 = a + b + c$ and $\|\mathbf{y}\|^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Cauchy–Schwarz gives $3 \leq \|\mathbf{x}\| \|\mathbf{y}\|$, and squaring both sides (both are ≥ 0) gives $9 \leq (a + b + c) (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})$. Equality holds exactly when $\mathbf{x}$ is a multiple of $\mathbf{y}$, that is, $\sqrt{a} = \frac{k}{\sqrt{a}}$, so $a = k$, and likewise $b = c = k$: exactly when $a = b = c$.