

Problem Sheet 1

Lectures 1 to 4. Part I covers Lectures 1 and 2, Part II covers Lectures 3 and 4, and Part III is practice on your own. Problems marked ► are solved in Recitation 1.

Part I. Vectors and their operations (Lectures 1 and 2)

A. Warm-up

Problem 1.

- (a) Draw, with tails at the origin, the vectors $(3, -1)$, $(-2, -2)$ and $(0, 4)$.
- (b) An arrow starts at $P = (1, 2)$ and ends at $Q = (4, -1)$. Which vector does it represent? Draw the same vector with its tail at the origin.

Problem 2. Let $\mathbf{u} = (1, -2, 4)$ and $\mathbf{v} = (3, 0, -1)$ in $\mathbb{R}^3$. Compute $\mathbf{u} + \mathbf{v}$, $3\mathbf{u}$, and $2\mathbf{u} - \mathbf{v}$.

Problem 3. ► Recitation

Find all real numbers a and b such that $\begin{bmatrix} a+b \\ a-b \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Is there a pair (a, b) with $\begin{bmatrix} a+b \\ a-b \\ 2a \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$?

B. Core

Problem 4. ► Recitation

A hiker records three legs of a walk as displacement vectors (in km, east and north): $(3, 1)$, then $(-1, 2)$, then $(2, -4)$.

- (a) What is the total displacement? Draw the three legs tip to tail and the total.
- (b) Which single displacement brings the hiker straight back to the start?
- (c) Does the total depend on the order in which the legs are walked? Explain using components.

Problem 5. ► Recitation

Find numbers x and y such that $x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$. Then describe your answer in the row picture and in the column picture. (As in Lecture 1: the row picture intersects the lines given by the equations; the column picture combines the columns to reach the right-hand side.)

Problem 6. ► Recitation

Colours on a screen are vectors (r, g, b) with each entry between 0 and 1. Let $\mathbf{c}_1 = (1, 0.4, 0)$ (orange) and $\mathbf{c}_2 = (0, 0.6, 1)$ (sky blue).

- (a) Compute the even mix $\frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2$. What colour is it?
- (b) For which numbers t between 0 and 1 is $t\mathbf{c}_1 + (1-t)\mathbf{c}_2$ a shade of grey, that is, a vector with all three entries equal?

C. Proof and concept

Problem 7. ► Recitation

Using only the definitions of vector addition and equality in $\mathbb{R}^n$, prove that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Problem 8. ► Recitation

Decide whether each statement is true for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Prove the true ones and give a counterexample for any false one.

- (a) If $2\mathbf{u} = 2\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.
- (b) If $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.
- (c) If $\mathbf{u}$ and $\mathbf{v}$ have the same first component, then $\mathbf{u} = \mathbf{v}$.

Part II. Linear combinations, span and the dot product (Lectures 3 and 4)

A. Warm-up

Problem 9. ► Recitation

Let $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 4)$ in $\mathbb{R}^3$.

- (a) Compute $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.
- (b) Find the cosine of the angle θ between $\mathbf{u}$ and $\mathbf{v}$. Is θ acute, right or obtuse?
- (c) Find the unit vector in the direction of $\mathbf{v}$.
- (d) For which number t is the vector $(1, t, 2)$ orthogonal to $\mathbf{u}$?

B. Core

Problem 10. ► Recitation

Let $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (-1, 3)$.

- (a) On one grid, draw $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, $2\mathbf{v} - \mathbf{w}$ and $-\mathbf{v} + \frac{1}{2}\mathbf{w}$, all with tails at the origin.
- (b) Find numbers x and y with $x\mathbf{v} + y\mathbf{w} = (-3, 5)$.
- (c) Show that every vector $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^2$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. What is $\text{Span}\{\mathbf{v}, \mathbf{w}\}$?
- (d) Would the answer to (c) change if $\mathbf{w}$ were replaced by $(-2, -2)$? Explain.

Problem 11. ► Recitation

- (a) Sketch the span of each set in $\mathbb{R}^2$: (i) $\{(2, -1)\}$; (ii) $\{(2, -1), (-4, 2)\}$; (iii) $\{(2, -1), (1, 1)\}$; (iv) $\{(0, 0)\}$.

- (b) In $\mathbb{R}^3$ let $\mathbf{a} = (1, 0, 2)$, $\mathbf{b} = (0, 1, -1)$, $\mathbf{c} = (2, 3, 1)$ and $\mathbf{d} = (1, 1, 2)$. Decide whether $\mathbf{c}$ and $\mathbf{d}$ lie in $\text{Span}\{\mathbf{a}, \mathbf{b}\}$. Then describe $\text{Span}\{\mathbf{a}, \mathbf{b}\}$ by an equation, and explain why $\text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is the same set.

Problem 12. ► Recitation

Let $A = (1, 0)$, $B = (4, 1)$ and $C = (0, 3)$.

- (a) Draw the triangle ABC .
- (b) Find the lengths of its three sides.
- (c) Use dot products to find its three angles.
- (d) Find the point D for which $ABDC$ is a square.

C. Proof and concept

Problem 13. ► Recitation

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

- (a) Prove the parallelogram law $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$, and say what it means for a parallelogram.
- (b) Prove that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$. What does this say about the diagonals of a rhombus?
- (c) True or false: if $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. Prove it or give a counterexample.

D. Challenge and application

Problem 14. ► Recitation

A search engine stores each document as a vector of word counts. With the four words (*vector*, *matrix*, *goal*, *team*), three short texts give

$$\mathbf{d}_1 = (4, 2, 0, 1), \quad \mathbf{d}_2 = (2, 2, 0, 0), \quad \mathbf{d}_3 = (0, 1, 3, 2).$$

The *cosine similarity* of two documents is the cosine of the angle between their vectors.

- (a) Compute the cosine similarity of each pair. Which two texts are most alike?
- (b) A fourth text is $\mathbf{d}_1$ copied out twice, so its vector is $2\mathbf{d}_1$. Show that its cosine similarity with any document $\mathbf{d}$ is the same as that of $\mathbf{d}_1$. Why is this a sensible property?
- (c) Explain why the cosine similarity of two word-count vectors (nonzero, with entries ≥ 0) always lies between 0 and 1, and when it equals 0.

Problem 15. ► Recitation

- (a) Apply the Cauchy–Schwarz inequality to $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{1} = (1, \dots, 1)$ to prove

$$(x_1 + x_2 + \dots + x_n)^2 \leq n(x_1^2 + x_2^2 + \dots + x_n^2).$$

When does equality hold?

- (b) Deduce that if $a + b + c = 1$, then $a^2 + b^2 + c^2 \geq \frac{1}{3}$. For which a, b, c is it equal to $\frac{1}{3}$?
- (c) Use the triangle inequality to prove that $\|\mathbf{u} - \mathbf{w}\| \leq \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\|$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. What does it say about the distances between three points?

Part III. Practice on your own (Lectures 1 to 4)

These problems are not solved in the recitation. Try them on your own; full solutions are in the solutions booklet.

A. Warm-up

Problem 16. Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (1, 4, 0)$.

- (a) Compute $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u} - 2\mathbf{v}$.
- (b) Compute $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.
- (c) Is $\mathbf{u}$ orthogonal to $(1, 2, 0)$?

B. Core

Problem 17. Let $\mathbf{a} = (2, 1)$ and $\mathbf{b} = (-1, 2)$ in $\mathbb{R}^2$.

- (a) Draw $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ on one grid.
- (b) Show that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and have the same length.
- (c) What shape do the points $\mathbf{0}$, $\mathbf{a}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{b}$ form? Find its area.
- (d) Find numbers x and y with $x\mathbf{a} + y\mathbf{b} = (0, 5)$.

Problem 18.

- (a) Find all vectors in $\mathbb{R}^2$ that are orthogonal to $(2, -3)$, and describe this set geometrically.
- (b) Which of these vectors have length $\sqrt{13}$?
- (c) Show that there is no number k for which $(1, k, 2)$ and $(k, -1, 3)$ are orthogonal.

C. Proof and concept

Problem 19. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

- (a) Prove: if $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{w}$ is orthogonal to every vector in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.
- (b) True or false: if $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$, then $\mathbf{u} \cdot \mathbf{w} = 0$. Prove it or give a counterexample.
- (c) Prove the Pythagorean theorem in $\mathbb{R}^n$: $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

D. Challenge and application

Problem 20. A shop sells three items at prices $\mathbf{p} = (2, 5, 3)$ (in euros), and a customer buys the quantities $\mathbf{q} = (4, 1, 6)$.

- (a) Explain why the total cost is $\mathbf{p} \cdot \mathbf{q}$, and compute it.
- (b) All prices rise by 10%. Write the new price vector and the new total cost, and explain why the cost also rises by exactly 10%.
- (c) (Challenge) For positive numbers a, b, c , apply the Cauchy–Schwarz inequality to $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}})$ to prove

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9.$$

When does equality hold?