

Problem Sheet 2: Solutions

Lectures 5 to 8: lines, planes and the cross product; fields; systems of linear equations; Gaussian elimination. Problems marked ► are solved in Recitation 2; the others are for practice on your own, with full solutions in the solutions booklet.

A. Warm-up

Problem 1. Let $\mathbf{u} = (2, 1, -1)$ and $\mathbf{v} = (1, 3, 2)$ in $\mathbb{R}^3$.

- (a) Compute $\mathbf{u} \times \mathbf{v}$ and check that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
- (b) Compute $\mathbf{v} \times \mathbf{u}$ without expanding a second time.
- (c) Find the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$.

Solution. (a) Component by component,

$$\mathbf{u} \times \mathbf{v} = (1 \cdot 2 - (-1) \cdot 3, (-1) \cdot 1 - 2 \cdot 2, 2 \cdot 3 - 1 \cdot 1) = (5, -5, 5).$$

Check: $(5, -5, 5) \cdot (2, 1, -1) = 10 - 5 - 5 = 0$ and $(5, -5, 5) \cdot (1, 3, 2) = 5 - 15 + 10 = 0$.

(b) The cross product is anticommutative, so $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = (-5, 5, -5)$.

(c) The area is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{25 + 25 + 25} = 5\sqrt{3} \approx 8.66$.

Problem 2.

- (a) In the field $\mathbb{Z}_7$, compute $5 + 4$, $5 \cdot 4$, -5 and 5^{-1} . Then solve $5x + 3 = 1$ in $\mathbb{Z}_7$.
- (b) Which of these are fields, with the given addition and multiplication? Give a one-line reason for each: $\mathbb{Z}_{11}$; $\mathbb{Z}_9$; the positive real numbers, with the usual operations; the set $\{0\}$, with $0 + 0 = 0$ and $0 \cdot 0 = 0$.

Solution. (a) $5 + 4 = 9 = 2$, $5 \cdot 4 = 20 = 6$ (since $20 = 2 \cdot 7 + 6$), $-5 = 2$ (since $5 + 2 = 7 = 0$), and $5^{-1} = 3$ (since $5 \cdot 3 = 15 = 1$). For the equation, add -3 to both sides: $5x = 1 - 3 = -2 = 5$. Multiply by $5^{-1} = 3$: $x = 5 \cdot 3 = 15 = 1$. Check: $5 \cdot 1 + 3 = 8 = 1$ in $\mathbb{Z}_7$.

(b) $\mathbb{Z}_{11}$ is a field: 11 is prime, and $\mathbb{Z}_n$ is a field exactly when n is prime (Lecture 6). $\mathbb{Z}_9$ is not a field: $3 \cdot 3 = 9 = 0$ with $3 \neq 0$, and a field has no zero divisors. The positive real numbers are not a field: there is no zero element in the set (no positive z has $1 + z = 1$), and no negatives either. $\{0\}$ is not a field: the axioms require a one element different from zero, and this set has only one element.

Problem 3.

- (a) Write the augmented matrix of the system (first put each equation in standard form, with the unknowns in the order x, y, z):

$$x + 3z = 2, \quad y = 2x + z, \quad 4z + 5 = y.$$

- (b) Write the system of equations, and the same system as one vector equation (the column picture of Lecture 7: a combination of the columns equal to the right-hand side), whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & -1 \end{array} \right]$.

Solution. (a) In standard form the equations are $x + 3z = 2$, $2x - y + z = 0$, $y - 4z = 5$. One row per equation and one column per unknown, with a 0 for every missing unknown:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -4 & 5 \end{array} \right].$$

(b) With unknowns x, y, z : $x - 2z = 3$ and $y + 4z = -1$. As a vector equation,

$$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Problem 4. Solve by back substitution:

$$x - y + 2z = 5, \quad 2y - z = 0, \quad 3z = 6.$$

Solution. The last equation gives $z = 2$. The second gives $2y = z = 2$, so $y = 1$. The first gives $x = 5 + y - 2z = 5 + 1 - 4 = 2$. So $(x, y, z) = (2, 1, 2)$. Check: $2 - 1 + 4 = 5$, $2 - 2 = 0$, $6 = 6$.

B. Core

Problem 5. ► Recitation

Let $P = (2, -1, 1)$ and $Q = (4, 0, -2)$.

- Write the line L through P and Q in parametric form.
- Does $R = (8, 2, -8)$ lie on L ? Does $S = (6, 1, -4)$?
- Where does L meet the plane $x - y + z = 10$?
- Show that L never meets the plane $x + y + z = 6$, and explain why using the plane's normal vector. How far is L from this plane?

Solution. (a) A direction vector is $\mathbf{d} = Q - P = (2, 1, -3)$, so

$$L: \quad \mathbf{x} = (2, -1, 1) + t(2, 1, -3) = (2 + 2t, -1 + t, 1 - 3t), \quad t \in \mathbb{R}.$$

(b) For R the second component forces $-1 + t = 2$, so $t = 3$, and then $\mathbf{x} = (8, 2, -8) = R$: R is on L . For S the second component forces $t = 2$, and then $\mathbf{x} = (6, 1, -5) \neq S$ (the third components differ), so S is not on L .

(c) Substitute the point of L into the equation of the plane: $(2 + 2t) - (-1 + t) + (1 - 3t) = 4 - 2t = 10$, so $t = -3$. The meeting point is $(2 - 6, -1 - 3, 1 + 9) = (-4, -4, 10)$. Check: $-4 + 4 + 10 = 10$.

(d) Substituting gives $(2 + 2t) + (-1 + t) + (1 - 3t) = 2$ for every t , and $2 \neq 6$. So no point of L satisfies the equation, and L misses the plane. The reason: the normal vector $\mathbf{n} = (1, 1, 1)$ has $\mathbf{n} \cdot \mathbf{d} = 2 + 1 - 3 = 0$, so the direction of L is orthogonal to the normal. That means L runs parallel to the plane: every point of L has $x + y + z = 2$, so it stays at the same distance from the plane. By the distance formula of Lecture 5, with the point P of L : the distance is $\frac{|\mathbf{n} \cdot P - 6|}{\|\mathbf{n}\|} = \frac{|2 - 6|}{\sqrt{3}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \approx 2.31$.

Problem 6. ► Recitation

Let $A = (1, 0, 1)$, $B = (2, 2, 1)$ and $C = (1, 2, 2)$.

- (a) Use the cross product to find a normal vector of the plane through A , B and C , and write the plane's equation in normal form.
- (b) Find the area of the triangle ABC .
- (c) Does $D = (0, 0, 2)$ lie in the same plane as A , B , C ? Does $E = (1, 1, 1)$?
- (d) Find the point of the plane closest to the origin, and the distance from the origin to the plane. (Hint: the shortest path runs along the normal.)

Solution. (a) Two directions in the plane are $\mathbf{u} = B - A = (1, 2, 0)$ and $\mathbf{v} = C - A = (0, 2, 1)$. Their cross product is orthogonal to both:

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = (2 \cdot 1 - 0 \cdot 2, 0 \cdot 0 - 1 \cdot 1, 1 \cdot 2 - 2 \cdot 0) = (2, -1, 2).$$

Check: $\mathbf{n} \cdot \mathbf{u} = 2 - 2 + 0 = 0$ and $\mathbf{n} \cdot \mathbf{v} = 0 - 2 + 2 = 0$. The plane is $\mathbf{n} \cdot (\mathbf{x} - A) = 0$, that is $2x - y + 2z = \mathbf{n} \cdot A = 2 + 0 + 2$, or

$$2x - y + 2z = 4.$$

Check with the other two points: B gives $4 - 2 + 2 = 4$, and C gives $2 - 2 + 4 = 4$.

(b) $\|\mathbf{n}\| = \sqrt{4 + 1 + 4} = 3$ is the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$, and the triangle is half of it: area = $\frac{3}{2}$.

(c) D : $0 - 0 + 4 = 4$, so D is in the plane. E : $2 - 1 + 2 = 3 \neq 4$, so E is not.

(d) The line through the origin along the normal is $t(2, -1, 2)$. It meets the plane when $2(2t) - (-t) + 2(2t) = 9t = 4$, so $t = \frac{4}{9}$, at the point $F = (\frac{8}{9}, -\frac{4}{9}, \frac{8}{9})$. The distance is $\|F\| = \frac{4}{9}\|\mathbf{n}\| = \frac{4}{9} \cdot 3 = \frac{4}{3}$. This agrees with the formula of Lecture 5: $\frac{|\mathbf{n} \cdot \mathbf{0} - 4|}{\|\mathbf{n}\|} = \frac{4}{3}$.

Problem 7. ► Recitation

Solve by Gaussian elimination, then check your answer:

$$x + y + z = 2, \quad 2x + 3y + z = -1, \quad x - y + 2z = 9.$$

Describe the solution in the row picture and in the column picture (Lecture 7: the row picture intersects the planes given by the equations; the column picture combines the columns to reach the right-hand side).

Solution. Start from the augmented matrix and clear the first column below the first pivot, then the second:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & -1 \\ 1 & -1 & 2 & 9 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & -2 & 1 & 7 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & -1 & -3 \end{array} \right].$$

Back substitution: $-z = -3$ gives $z = 3$; then $y - z = -5$ gives $y = -2$; then $x + y + z = 2$ gives $x = 2 + 2 - 3 = 1$. So $(x, y, z) = (1, -2, 3)$. Check: $1 - 2 + 3 = 2$, $2 - 6 + 3 = -1$, $1 + 2 + 6 = 9$.

Row picture: each equation is a plane in $\mathbb{R}^3$, and the three planes meet in exactly one point, $(1, -2, 3)$. Column picture: $1 \cdot (1, 2, 1) - 2 \cdot (1, 3, -1) + 3 \cdot (1, 1, 2) = (1 - 2 + 3, 2 - 6 + 3, 1 + 2 + 6) = (2, -1, 9)$, so this combination of the three columns reaches the right-hand side.

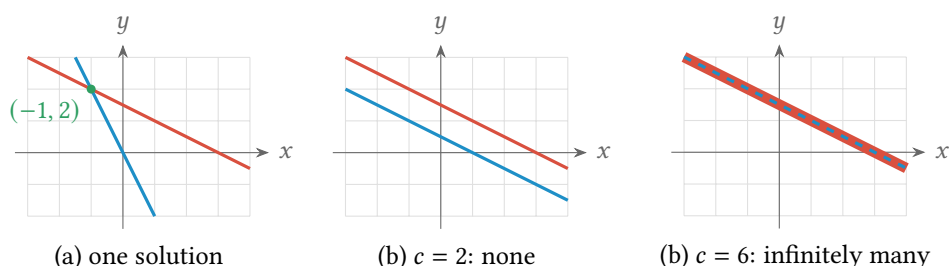
Problem 8. Consider the system $x + 2y = 3$, $2x + ky = c$.

- (a) For $k = 1$ and $c = 0$, solve the system and draw its row picture (Lecture 7: each equation is a line, and the solutions are the points the lines share).

- (b) Now let $k = 4$. Draw the row picture for $c = 2$ and for $c = 6$, and say how many solutions each system has.
- (c) Still with $k = 4$, draw the column picture: the two columns and the right-hand side $(3, c)$ for $c = 2$ and $c = 6$. Use it to explain for which c the system has a solution.

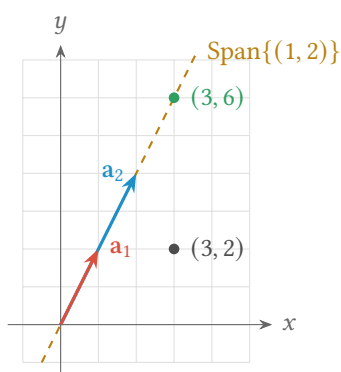
Solution. (a) $R_2 \rightarrow R_2 - 2R_1$ turns $2x + y = 0$ into $-3y = -6$, so $y = 2$, and then $x = 3 - 2y = -1$. Check: $-1 + 4 = 3$ and $-2 + 2 = 0$. The two lines cross at the single point $(-1, 2)$.

(b) With $k = 4$, the replacement $R_2 \rightarrow R_2 - 2R_1$ gives $0 = c - 6$. For $c = 2$ this says $0 = -4$: no solution. In the row picture, $2x + 4y = 2$ is the line $x + 2y = 1$, parallel to $x + 2y = 3$, and parallel lines never meet. For $c = 6$ it says $0 = 0$: the second equation is twice the first, both are the same line, and every point of it is a solution, so there are infinitely many.



Red: $x + 2y = 3$. Blue: the second equation (in the last picture it lies on top of the red line).

(c) The columns are $\mathbf{a}_1 = (1, 2)$ and $\mathbf{a}_2 = (2, 4) = 2\mathbf{a}_1$. Both lie on the line $\text{Span}\{(1, 2)\}$ through the origin, so every combination $x\mathbf{a}_1 + y\mathbf{a}_2 = (x + 2y)\mathbf{a}_1$ stays on that line. The system has a solution exactly when $(3, c)$ lies on this line, that is when $c = 2 \cdot 3 = 6$. For $c = 6$, for example $3\mathbf{a}_1 + 0\mathbf{a}_2$ and $1\mathbf{a}_1 + 1\mathbf{a}_2$ both reach $(3, 6)$. For $c = 2$ the point $(3, 2)$ is off the line, and no combination reaches it.



Problem 9. ► Recitation

Solve the system

$$2x + y = 1, \quad x + 3y = 2$$

- (a) over $\mathbb{R}$; (b) over $\mathbb{Z}_7$; (c) over $\mathbb{Z}_5$.

Use Gaussian elimination each time, and explain why the three answers differ.

Solution. In all three fields, swap the rows so that the first pivot is 1, then subtract twice the new first row from the second:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 3 & 2 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 1 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -5 & -3 \end{array} \right].$$

So far only integers were added and multiplied, so this is correct in every field. The fields differ in the last step.

(a) Over $\mathbb{R}$: $-5y = -3$ gives $y = \frac{3}{5}$, and $x = 2 - 3y = 2 - \frac{9}{5} = \frac{1}{5}$. So $(x, y) = (\frac{1}{5}, \frac{3}{5})$. Check: $\frac{2}{5} + \frac{3}{5} = 1$ and $\frac{1}{5} + \frac{9}{5} = 2$.

(b) Over $\mathbb{Z}_7$: $-5 = 2$ and $-3 = 4$, so the last row says $2y = 4$. Multiply by $2^{-1} = 4$ (since $2 \cdot 4 = 8 = 1$): $y = 16 = 2$. Then $x = 2 - 3 \cdot 2 = -4 = 3$. So $(x, y) = (3, 2)$. Check: $2 \cdot 3 + 2 = 8 = 1$ and $3 + 3 \cdot 2 = 9 = 2$ in $\mathbb{Z}_7$.

(c) Over $\mathbb{Z}_5$: $-5 = 0$ and $-3 = 2$, so the last row says $0 \cdot y = 2$, that is $0 = 2$. No y satisfies this, so the system has no solution over $\mathbb{Z}_5$.

Why they differ: elimination is the same sequence of operations in every field, but whether the second pivot -5 is zero depends on the field. In $\mathbb{R}$ and $\mathbb{Z}_7$ it is nonzero, so we can divide by it; in $\mathbb{Z}_5$ it is 0, the two lines (in the row picture over $\mathbb{Z}_5$) are “parallel”, and the equations contradict each other.

Problem 10.

(a) Solve by Gaussian elimination:

$$x + y + z = 2, \quad 2x + 2y + 3z = 5, \quad x + 2y + 2z = 2.$$

(b) Show by Gaussian elimination that this system has no solution:

$$x + 2y - z = 1, \quad 2x + 3y + z = 4, \quad 3x + 5y = 2.$$

(c) In (b), to which number must the last constant 2 be changed so that the system has a solution?

Solution. (a) After clearing the first column, the entry in the second pivot position is 0, so we swap in the row below:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 5 \\ 1 & 2 & 2 & 2 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right].$$

Back substitution: $z = 1$, $y = 0 - z = -1$, $x = 2 - y - z = 2 + 1 - 1 = 2$. So $(x, y, z) = (2, -1, 1)$. Check: $2 - 1 + 1 = 2$, $4 - 2 + 3 = 5$, $2 - 2 + 2 = 2$.

(b)

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 5 & 0 & 2 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - 3R_1]{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & -1 & 3 & -1 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & -3 \end{array} \right].$$

The last row is the equation $0x + 0y + 0z = -3$, that is $0 = -3$, which no (x, y, z) satisfies. Row operations do not change the solution set (Lecture 8), so the original system has no solution either. In the row picture, no two of the three planes are parallel (their normal vectors $(1, 2, -1)$, $(2, 3, 1)$, $(3, 5, 0)$ are not multiples of one another), yet they have no common point: each pair meets in a line, and the three lines run side by side, like the edges of a tent (Lecture 7).

(c) With last constant b , the same steps give the last entry $(b - 3) - 2 = b - 5$. The system is consistent only when this is 0, so the constant must be 5. (Then the left side of the third equation is the sum of the first two, and so is its right side: $1 + 4 = 5$. The last row becomes $0 = 0$, and there are infinitely many solutions, which Lecture 9 describes.)

C. Proof and concept

Problem 11. ► Recitation

Let $\mathbb{F}$ be a field. You may use the three facts proved in Lecture 6: $0 \cdot a = 0$, $(-1)a = -a$, and if $ab = 0$ then $a = 0$ or $b = 0$.

- (a) Prove that $(-a)(-b) = ab$ for all $a, b \in \mathbb{F}$.
- (b) Prove that the only solutions of $x^2 = 1$ in $\mathbb{F}$ are $x = 1$ and $x = -1$.
- (c) Find all solutions of $x^2 = 1$ in $\mathbb{Z}_8$. Explain why this shows that $\mathbb{Z}_8$ is not a field.

Solution. (a) By the second fact, $-a = (-1)a$ and $-b = (-1)b$. Since multiplication is associative and commutative, we may regroup the four factors:

$$(-a)(-b) = ((-1)a)((-1)b) = ((-1)(-1))(ab).$$

Now $(-1)(-1) = -(-1)$, by the second fact with $a = -1$. And $-(-1) = 1$: the negative of -1 is the element that gives 0 when added to -1 , and $1 + (-1) = 0$, so it is 1 (an element has only one negative). Hence $(-a)(-b) = 1 \cdot (ab) = ab$. $\square$

(b) First, both are solutions: $1 \cdot 1 = 1$, and $(-1)(-1) = 1 \cdot 1 = 1$ by part (a). Now suppose $x^2 = 1$. By the distributive law and $(-1)x = -x$,

$$(x-1)(x+1) = x \cdot x + x \cdot 1 + (-1)x + (-1) \cdot 1 = x^2 + x - x - 1 = x^2 - 1 = 0.$$

A field has no zero divisors, so $x-1 = 0$ or $x+1 = 0$, that is $x = 1$ or $x = -1$. $\square$ (In $\mathbb{Z}_2$, $-1 = 1$, so there the two solutions coincide.)

(c) Squaring each element of $\mathbb{Z}_8$:

$$0^2 = 0, 1^2 = 1, 2^2 = 4, 3^2 = 9 = 1, 4^2 = 16 = 0, 5^2 = 25 = 1, 6^2 = 36 = 4, 7^2 = 49 = 1.$$

So $x^2 = 1$ has four solutions, $x = 1, 3, 5, 7$. In $\mathbb{Z}_8$ we have $-1 = 7$, so 3 and 5 are neither 1 nor -1 . By part (b) this cannot happen in a field, so $\mathbb{Z}_8$ is not a field. (The proof of (b) breaks down exactly where it used “no zero divisors”: for $x = 3$, $(x-1)(x+1) = 2 \cdot 4 = 8 = 0$ although neither factor is 0.)

Problem 12. Decide whether each statement holds for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Prove the true ones from the component formula for the cross product, and give a counterexample for each false one.

- (a) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
- (b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
- (c) If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
- (d) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

Solution. Recall $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$, and write $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$.

(a) False. For parallel vectors such as $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (2, 4, 6)$: $\mathbf{u} \times \mathbf{v} = (2 \cdot 6 - 3 \cdot 4, 3 \cdot 2 - 1 \cdot 6, 1 \cdot 4 - 2 \cdot 2) = (0, 0, 0)$, although neither vector is $\mathbf{0}$. (By Lecture 5 the cross product is $\mathbf{0}$ exactly when the vectors are parallel.)

(b) True. The first component of the left side is

$$u_2(v_3 + w_3) - u_3(v_2 + w_2) = (u_2v_3 - u_3v_2) + (u_2w_3 - u_3w_2),$$

by the distributive law for real numbers, and the right-hand side is exactly the sum of the first components of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{w}$. The second and third components work the same way, with the indices 3, 1 and 1, 2 in place of 2, 3. So the two vectors agree in every component. $\square$

(c) False. Take $\mathbf{u} = \mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_2$ and $\mathbf{w} = \mathbf{e}_1 + \mathbf{e}_2 = (1, 1, 0)$. Then $\mathbf{e}_1 \times \mathbf{e}_2 = (0, 0, 1)$ and $\mathbf{e}_1 \times (1, 1, 0) = (0 \cdot 0 - 0 \cdot 1, 0 \cdot 1 - 1 \cdot 0, 1 \cdot 1 - 0 \cdot 1) = (0, 0, 1)$, but $\mathbf{v} \neq \mathbf{w}$. We cannot “cancel” $\mathbf{u}$: by (b) and the rule $\mathbf{u} \times (c\mathbf{w}) = c(\mathbf{u} \times \mathbf{w})$, $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ only says $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$, that is, $\mathbf{v} - \mathbf{w}$ is parallel to $\mathbf{u}$.

(d) False. Take $\mathbf{u} = \mathbf{v} = \mathbf{e}_1$ and $\mathbf{w} = \mathbf{e}_2$. Then $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}$, so the left side is $\mathbf{0} \times \mathbf{e}_2 = \mathbf{0}$. On the right, $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ and $\mathbf{e}_1 \times \mathbf{e}_3 = (0 \cdot 1 - 0 \cdot 0, 0 \cdot 0 - 1 \cdot 1, 1 \cdot 0 - 0 \cdot 0) = (0, -1, 0)$, which is not $\mathbf{0}$. The cross product is not associative.

Problem 13. ► Recitation

True or false? Prove each true statement and give a counterexample for each false one.

- (a) Performing two replacements at the same time, $R_1 \rightarrow R_1 + R_2$ and $R_2 \rightarrow R_2 + R_1$ (both computed from the old rows), never changes the solution set of a system.
- (b) If a system of linear equations over $\mathbb{R}$ in three unknowns has the solutions $(1, 1, 1)$ and $(2, 2, 2)$, then $(5, 5, 5)$ is also a solution.
- (c) A system of linear equations over $\mathbb{Z}_3$ can have exactly three solutions.

Solution. (a) False. Take the system $x = 1, y = 2$, whose only solution is $(1, 2)$. Both new rows are computed from the old ones, so both become $x + y = 3$. The new system $x + y = 3, x + y = 3$ has, for example, the solution $(0, 3)$, which does not solve the old one. Solutions were gained: the information in the old rows was lost, and the step cannot be undone. Each row operation of Lecture 8 can be undone by another row operation: a swap by the same swap, a scaling $R_i \rightarrow cR_i$ ($c \neq 0$) by $R_i \rightarrow c^{-1}R_i$, and a replacement $R_i \rightarrow R_i + cR_j$ by $R_i \rightarrow R_i - cR_j$, because row j is unchanged. The simultaneous pair cannot be undone.

(b) True. Write equation i as $\mathbf{r}_i \cdot \mathbf{x} = b_i$, where $\mathbf{r}_i$ is the i th row of coefficients (Lecture 7), and let $\mathbf{s} = (1, 1, 1)$, $\mathbf{t} = (2, 2, 2)$. Then $(5, 5, 5) = \mathbf{s} + 4(\mathbf{t} - \mathbf{s})$, and for every i ,

$$\mathbf{r}_i \cdot (\mathbf{s} + 4(\mathbf{t} - \mathbf{s})) = \mathbf{r}_i \cdot \mathbf{s} + 4(\mathbf{r}_i \cdot \mathbf{t} - \mathbf{r}_i \cdot \mathbf{s}) = b_i + 4(b_i - b_i) = b_i.$$

So $(5, 5, 5)$ solves every equation. (Every point $\mathbf{s} + c(\mathbf{t} - \mathbf{s})$ of the line through the two solutions is a solution.) $\square$

(c) True. Over $\mathbb{Z}_3$, the single equation $x + y = 1$ has exactly three solutions: each of the three values of x fixes $y = 1 - x$, giving $(0, 1)$, $(1, 0)$ and $(2, 2)$ (check: $2 + 2 = 4 = 1$). The rule “no solution, one, or infinitely many” (Lecture 7) needs infinitely many scalars c ; in $\mathbb{Z}_3$ the line through two solutions has only three points.

D. Challenge and application

Problem 14. A quantity y is measured at three times: $y = 2$ at $x = 1$, $y = 3$ at $x = 2$, and $y = 6$ at $x = 3$.

- (a) Find the parabola $y = a + bx + cx^2$ through the three data points by setting up and solving a linear system for a, b, c . Sketch the parabola and the data points.
- (b) Show that no straight line $y = a + bx$ passes through all three points. What does elimination show?

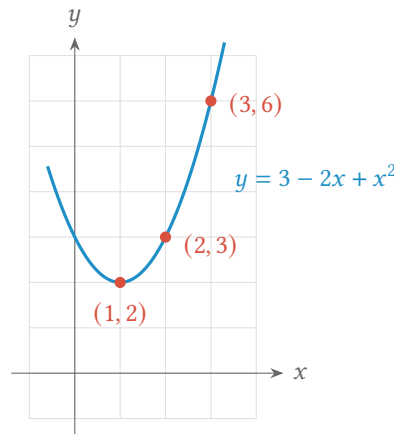
- (c) Why must the three x -values be different for (a) to work? What goes wrong in the system if two of them are equal?

Solution. (a) Each data point gives one linear equation in the unknowns a, b, c :

$$a + b + c = 2, \quad a + 2b + 4c = 3, \quad a + 3b + 9c = 6.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 6 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 4 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 2 \end{array} \right].$$

So $c = 1$, $b = 1 - 3c = -2$, $a = 2 - b - c = 3$, and the parabola is $y = 3 - 2x + x^2$. Check: $3 - 2 + 1 = 2$, $3 - 4 + 4 = 3$, $3 - 6 + 9 = 6$.



The lowest point of the parabola is $(1, 2)$, the first data point.

(b) A line needs $a + b = 2$, $a + 2b = 3$, $a + 3b = 6$. Elimination: $R_2 - R_1$ gives $b = 1$ and $R_3 - R_1$ gives $2b = 4$; then $R_3 - 2R_2$ leaves $0 = 2$. The system is inconsistent, so no line fits all three points: the slope from the first point to the second is 1, and from the second to the third it is 3. (Fitting the best line through data that no line fits exactly is the method of least squares, in Week 12.)

(c) If two x -values are equal, say both are x_0 , then two rows of the system are $1, x_0, x_0^2$ on the left. If their y -values differ, subtracting one row from the other gives $0 = (\text{nonzero})$, so there is no solution (no function takes two values at x_0). If their y -values agree, the row becomes $0 = 0$ and we really have only two equations for three unknowns, so the parabola is not pinned down.

Problem 15. An ISBN-10 is a list of ten digits $x_1, \dots, x_{10}$ (the last may be 10, printed X) chosen so that

$$1x_1 + 2x_2 + 3x_3 + \dots + 10x_{10} = 0 \quad \text{in } \mathbb{Z}_{11}.$$

- The first nine digits of a book code are 0, 8, 5, 3, 1, 2, 6, 7, 4. Find the check digit x_{10} .
- Suppose exactly one digit is copied wrongly. Use the fact that $\mathbb{Z}_{11}$ is a field, which has no zero divisors (Lecture 6), to show that the condition always fails, so the error is detected.
- Show that swapping two different neighbouring digits is also always detected.
- A shop uses the same rule but in $\mathbb{Z}_{10}$ (arithmetic mod 10) instead. Give a single-digit error that this rule misses.

Solution. (a) $1 \cdot 0 + 2 \cdot 8 + 3 \cdot 5 + 4 \cdot 3 + 5 \cdot 1 + 6 \cdot 2 + 7 \cdot 6 + 8 \cdot 7 + 9 \cdot 4 = 0 + 16 + 15 + 12 + 5 + 12 + 42 + 56 + 36 = 194$, and $194 = 17 \cdot 11 + 7$, so the sum is 7 in $\mathbb{Z}_{11}$. We need $7 + 10x_{10} = 0$. Since $10 = -1$ in $\mathbb{Z}_{11}$, this says $7 - x_{10} = 0$, so $x_{10} = 7$. Check: $194 + 70 = 264 = 24 \cdot 11$.

(b) Suppose digit x_i is replaced by $x_i + e$ with $e \neq 0$; since both are digits, $-10 \leq e \leq 10$, so e is not a multiple of 11 and $e \neq 0$ in $\mathbb{Z}_{11}$. The weighted sum changes by $i \cdot e$. Here $i \in \{1, \dots, 10\}$, so $i \neq 0$ in $\mathbb{Z}_{11}$ as well. Because $\mathbb{Z}_{11}$ is a field (11 is prime), it has no zero divisors: a product of two nonzero elements is nonzero, so $i e \neq 0$. The new sum is $0 + ie \neq 0$, and the error is detected.

(c) Swap x_i and x_{i+1} with $x_i \neq x_{i+1}$. The terms $ix_i + (i+1)x_{i+1}$ become $ix_{i+1} + (i+1)x_i$, a change of $x_i - x_{i+1}$. This is a nonzero number between -10 and 10 , so it is nonzero in $\mathbb{Z}_{11}$, and the sum is no longer 0.

(d) In $\mathbb{Z}_{10}$, $2 \cdot 5 = 10 = 0$. So if the second digit changes by 5 (say from 0 to 5), the sum changes by $2 \cdot 5 = 0$ and the error goes unnoticed. The rule fails because $\mathbb{Z}_{10}$ has zero divisors: it is not a field.