

Problem Sheet 2

Lectures 5 to 8: lines, planes and the cross product; fields; systems of linear equations; Gaussian elimination. Problems marked ► are solved in Recitation 2; the others are for practice on your own, with full solutions in the solutions booklet.

A. Warm-up

Problem 1. Let $\mathbf{u} = (2, 1, -1)$ and $\mathbf{v} = (1, 3, 2)$ in $\mathbb{R}^3$.

- (a) Compute $\mathbf{u} \times \mathbf{v}$ and check that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
- (b) Compute $\mathbf{v} \times \mathbf{u}$ without expanding a second time.
- (c) Find the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$.

Problem 2.

- (a) In the field $\mathbb{Z}_7$, compute $5 + 4$, $5 \cdot 4$, -5 and 5^{-1} . Then solve $5x + 3 = 1$ in $\mathbb{Z}_7$.
- (b) Which of these are fields, with the given addition and multiplication? Give a one-line reason for each: $\mathbb{Z}_{11}$; $\mathbb{Z}_9$; the positive real numbers, with the usual operations; the set $\{0\}$, with $0 + 0 = 0$ and $0 \cdot 0 = 0$.

Problem 3.

- (a) Write the augmented matrix of the system (first put each equation in standard form, with the unknowns in the order x, y, z):

$$x + 3z = 2, \quad y = 2x + z, \quad 4z + 5 = y.$$

- (b) Write the system of equations, and the same system as one vector equation (the column picture of Lecture 7: a combination of the columns equal to the right-hand side), whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & -1 \end{array} \right]$.

Problem 4. Solve by back substitution:

$$x - y + 2z = 5, \quad 2y - z = 0, \quad 3z = 6.$$

B. Core

Problem 5. ► Recitation

Let $P = (2, -1, 1)$ and $Q = (4, 0, -2)$.

- (a) Write the line L through P and Q in parametric form.
- (b) Does $R = (8, 2, -8)$ lie on L ? Does $S = (6, 1, -4)$?
- (c) Where does L meet the plane $x - y + z = 10$?

- (d) Show that L never meets the plane $x + y + z = 6$, and explain why using the plane's normal vector. How far is L from this plane?

Problem 6. ► Recitation

Let $A = (1, 0, 1)$, $B = (2, 2, 1)$ and $C = (1, 2, 2)$.

- Use the cross product to find a normal vector of the plane through A , B and C , and write the plane's equation in normal form.
- Find the area of the triangle ABC .
- Does $D = (0, 0, 2)$ lie in the same plane as A , B , C ? Does $E = (1, 1, 1)$?
- Find the point of the plane closest to the origin, and the distance from the origin to the plane. (Hint: the shortest path runs along the normal.)

Problem 7. ► Recitation

Solve by Gaussian elimination, then check your answer:

$$x + y + z = 2, \quad 2x + 3y + z = -1, \quad x - y + 2z = 9.$$

Describe the solution in the row picture and in the column picture (Lecture 7: the row picture intersects the planes given by the equations; the column picture combines the columns to reach the right-hand side).

Problem 8. Consider the system $x + 2y = 3$, $2x + ky = c$.

- For $k = 1$ and $c = 0$, solve the system and draw its row picture (Lecture 7: each equation is a line, and the solutions are the points the lines share).
- Now let $k = 4$. Draw the row picture for $c = 2$ and for $c = 6$, and say how many solutions each system has.
- Still with $k = 4$, draw the column picture: the two columns and the right-hand side $(3, c)$ for $c = 2$ and $c = 6$. Use it to explain for which c the system has a solution.

Problem 9. ► Recitation

Solve the system

$$2x + y = 1, \quad x + 3y = 2$$

- over $\mathbb{R}$; (b) over $\mathbb{Z}_7$; (c) over $\mathbb{Z}_5$.

Use Gaussian elimination each time, and explain why the three answers differ.

Problem 10.

- Solve by Gaussian elimination:

$$x + y + z = 2, \quad 2x + 2y + 3z = 5, \quad x + 2y + 2z = 2.$$

- Show by Gaussian elimination that this system has no solution:

$$x + 2y - z = 1, \quad 2x + 3y + z = 4, \quad 3x + 5y = 2.$$

- In (b), to which number must the last constant 2 be changed so that the system has a solution?

C. Proof and concept

Problem 11. ► Recitation

Let $\mathbb{F}$ be a field. You may use the three facts proved in Lecture 6: $0 \cdot a = 0$, $(-1)a = -a$, and if $ab = 0$ then $a = 0$ or $b = 0$.

- (a) Prove that $(-a)(-b) = ab$ for all $a, b \in \mathbb{F}$.
- (b) Prove that the only solutions of $x^2 = 1$ in $\mathbb{F}$ are $x = 1$ and $x = -1$.
- (c) Find all solutions of $x^2 = 1$ in $\mathbb{Z}_8$. Explain why this shows that $\mathbb{Z}_8$ is not a field.

Problem 12. Decide whether each statement holds for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Prove the true ones from the component formula for the cross product, and give a counterexample for each false one.

- (a) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
- (b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
- (c) If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
- (d) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

Problem 13. ► Recitation

True or false? Prove each true statement and give a counterexample for each false one.

- (a) Performing two replacements at the same time, $R_1 \rightarrow R_1 + R_2$ and $R_2 \rightarrow R_2 + R_1$ (both computed from the old rows), never changes the solution set of a system.
- (b) If a system of linear equations over $\mathbb{R}$ in three unknowns has the solutions $(1, 1, 1)$ and $(2, 2, 2)$, then $(5, 5, 5)$ is also a solution.
- (c) A system of linear equations over $\mathbb{Z}_3$ can have exactly three solutions.

D. Challenge and application

Problem 14. A quantity y is measured at three times: $y = 2$ at $x = 1$, $y = 3$ at $x = 2$, and $y = 6$ at $x = 3$.

- (a) Find the parabola $y = a + bx + cx^2$ through the three data points by setting up and solving a linear system for a, b, c . Sketch the parabola and the data points.
- (b) Show that no straight line $y = a + bx$ passes through all three points. What does elimination show?
- (c) Why must the three x -values be different for (a) to work? What goes wrong in the system if two of them are equal?

Problem 15. An ISBN-10 is a list of ten digits $x_1, \dots, x_{10}$ (the last may be 10, printed X) chosen so that

$$1x_1 + 2x_2 + 3x_3 + \dots + 10x_{10} = 0 \quad \text{in } \mathbb{Z}_{11}.$$

- (a) The first nine digits of a book code are 0, 8, 5, 3, 1, 2, 6, 7, 4. Find the check digit x_{10} .

- (b) Suppose exactly one digit is copied wrongly. Use the fact that $\mathbb{Z}_{11}$ is a field, which has no zero divisors (Lecture 6), to show that the condition always fails, so the error is detected.
- (c) Show that swapping two different neighbouring digits is also always detected.
- (d) A shop uses the same rule but in $\mathbb{Z}_{10}$ (arithmetic mod 10) instead. Give a single-digit error that this rule misses.