

Problem Sheet 3: Solutions

Lectures 9 to 12: reduced row echelon form; solution sets; linear independence; the matrix-vector product. Problems marked ► are solved in Recitation 3; the others are for practice on your own, with full solutions in the solutions booklet.

A. Warm-up

Problem 1. Recall (Lecture 9): a matrix is in *reduced row echelon form* (RREF) when it is in row echelon form, every pivot is 1, and every other entry in a pivot column is 0. For each matrix, say whether it is in RREF. If it is not, name a condition that fails. If it is, name its pivot columns.

$$\begin{array}{lll} \text{(a)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} & \text{(b)} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \text{(c)} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \\ \text{(d)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \text{(e)} \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{bmatrix} & \end{array}$$

Then bring (c) and (e) to RREF.

Solution. (a) In RREF; pivot columns 1 and 2. (b) In RREF; pivot columns 1, 3 and 4. (The 3 in column 2 is allowed: column 2 has no pivot.) (c) Not in RREF: it is in row echelon form, but the pivot column 2 has the nonzero entry 2 above its pivot. (d) Not in RREF, not even in row echelon form: the zero row is above a nonzero row. (e) Not in RREF: the pivot in row 2 is 2, not 1.

To reduce (c), clear the entry above the second pivot: $R_1 \rightarrow R_1 - 2R_2$ gives $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 3 \end{bmatrix}$ (in column 3, $0 - 2 \cdot 3 = -6$). To reduce (e), scale: $R_2 \rightarrow \frac{1}{2}R_2$ gives $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Column 3 now has a 0 above its pivot already, so this is the RREF.

Problem 2. Recall (Lecture 12): if A has columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ and $\mathbf{x} = (x_1, \dots, x_n)$, then $A\mathbf{x} = x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n$.

- Compute $A\mathbf{x}$ for $A = \begin{bmatrix} 3 & 0 & -2 \\ 1 & 4 & 1 \end{bmatrix}$ and $\mathbf{x} = (1, 2, -1)$, first as a combination of the columns, then by the row rule (entry i of $A\mathbf{x}$ is row i of A times $\mathbf{x}$, entry by entry and added up). Check that the two answers agree.
- Explain why $A\mathbf{y}$ is not defined for $\mathbf{y} = (1, 2)$.
- Let $B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{x} = (1, 2)$. Compute $B\mathbf{x}$, and draw the column picture: the columns $\mathbf{b}_1, \mathbf{b}_2$ of B , the combination $1\mathbf{b}_1 + 2\mathbf{b}_2$ tip to tail, and the result.

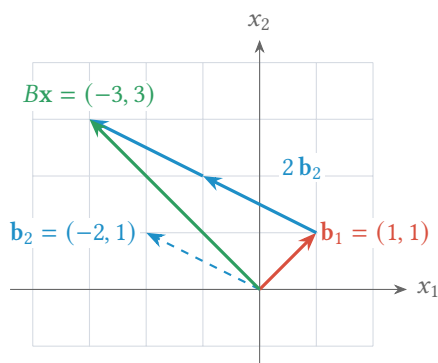
Solution. (a) As a combination of the columns:

$$A\mathbf{x} = 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 4 \end{bmatrix} - 1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 + 0 + 2 \\ 1 + 8 - 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

By the row rule: the first entry is $3 \cdot 1 + 0 \cdot 2 + (-2)(-1) = 3 + 0 + 2 = 5$, and the second is $1 \cdot 1 + 4 \cdot 2 + 1 \cdot (-1) = 1 + 8 - 1 = 8$. The same vector, $(5, 8)$.

(b) A has three columns, so $A\mathbf{y}$ needs three weights, one for each column; $\mathbf{y}$ has only two entries.

(c) $B\mathbf{x} = 1 \cdot (1, 1) + 2 \cdot (-2, 1) = (1 - 4, 1 + 2) = (-3, 3)$.



Red: the first column, taken once. Blue: the second column, taken twice, starting at the tip of the first (dashed: $\mathbf{b}_2$ itself from the origin). Green: the result $B\mathbf{x}$.

Problem 3.

- Write the system $3x_1 - x_2 + 2x_3 = 4$, $x_1 + 5x_3 = -1$ as a vector equation (a combination of the columns equal to the right-hand side) and as a matrix equation $A\mathbf{x} = \mathbf{b}$.
- Write the vector equation $x_1(1, 2, 0) + x_2(-1, 0, 3) = (4, 1, 1)$ as a system of equations and as $A\mathbf{x} = \mathbf{b}$.
- If A is a 4×3 matrix and $A\mathbf{x} = \mathbf{b}$, how many entries do $\mathbf{x}$ and $\mathbf{b}$ have?

Solution. (a) The columns are the coefficients of each unknown (a missing unknown has coefficient 0):

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

(b) One equation per entry: $x_1 - x_2 = 4$, $2x_1 = 1$, $3x_2 = 1$. As a matrix equation,

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

(c) $\mathbf{x}$ has one entry per column, so 3; $\mathbf{b}$ has one entry per row, so 4.

Problem 4. Decide by inspection whether each list of vectors is linearly independent, with a one-line reason.

- $(1, -2)$, $(-3, 6)$ in $\mathbb{R}^2$.
- $(1, 0, 2)$, $(0, 0, 0)$, $(3, 1, 1)$ in $\mathbb{R}^3$.
- $(2, 1)$, $(1, 3)$, $(4, -3)$ in $\mathbb{R}^2$. (Also find a dependence relation.)
- $(1, 1, 0)$, $(0, 1, 1)$ in $\mathbb{R}^3$.

Solution. (a) Dependent: $(-3, 6) = -3(1, -2)$, so $3(1, -2) + 1(-3, 6) = \mathbf{0}$.

(b) Dependent: the list contains the zero vector, and $0(1, 0, 2) + 1(0, 0, 0) + 0(3, 1, 1) = \mathbf{0}$ is a relation with a nonzero weight.

(c) Dependent: three vectors in $\mathbb{R}^2$ are always dependent, because the homogeneous system for the weights has 2 equations and 3 unknowns, so it has a free variable (Lecture 11). A relation: we need $2c_1 + c_2 + 4c_3 = 0$ and $c_1 + 3c_2 - 3c_3 = 0$. Take $c_3 = 1$. The second equation gives $c_1 = 3 - 3c_2$, and then the first gives $6 - 6c_2 + c_2 + 4 = 0$, so $c_2 = 2$ and $c_1 = -3$. Check: $-3(2, 1) + 2(1, 3) + (4, -3) = (-6 + 2 + 4, -3 + 6 - 3) = (0, 0)$.

(d) Independent: two vectors are dependent only when one is a multiple of the other (Lecture 11), and $(0, 1, 1)$ is not a multiple of $(1, 1, 0)$ (the first entries would force the multiple 0, but then the second entry would be 0, not 1); and $(1, 1, 0)$ is not a multiple of $(0, 1, 1)$, since every multiple of $(0, 1, 1)$ has first entry 0.

B. Core

Problem 5. ► Recitation

Consider the system

$$x_1 + x_2 + x_3 + 2x_4 = 2, \quad 2x_1 + x_2 + 3x_3 + x_4 = 6, \quad x_1 - x_2 + 3x_3 - 4x_4 = 6.$$

- Row reduce the augmented matrix to RREF. Which are the pivot columns, and which variables are free?
- Write the general solution in parametric vector form, $\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}$.
- Check that $\mathbf{p}$ solves the system and that $\mathbf{v}$ and $\mathbf{w}$ solve the homogeneous system with the same coefficients.
- Find the solution with $x_1 = 0$ and $x_2 = 0$.

Solution. (a) Clear the first column, then the second:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 3 & 1 & 6 \\ 1 & -1 & 3 & -4 & 6 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 2 \\ 0 & -1 & 1 & -3 & 2 \\ 0 & -2 & 2 & -6 & 4 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 2 \\ 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

This is a row echelon form. For the RREF, make the second pivot 1 and clear the entry above it:

$$\xrightarrow{R_2 \rightarrow -R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 4 \\ 0 & 1 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The pivot columns are 1 and 2, so x_1 and x_2 are pivot variables and x_3, x_4 are free. The last column is not a pivot column (the zero row says $0 = 0$), so the system is consistent (Lecture 10).

(b) The RREF says $x_1 + 2x_3 - x_4 = 4$ and $x_2 - x_3 + 3x_4 = -2$. Set $x_3 = s$ and $x_4 = t$; then $x_1 = 4 - 2s + t$ and $x_2 = -2 + s - 3t$:

$$\mathbf{x} = \begin{bmatrix} 4 - 2s + t \\ -2 + s - 3t \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ -2 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{p}} + s \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}} + t \underbrace{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}}, \quad s, t \in \mathbb{R}.$$

(c) $\mathbf{p} = (4, -2, 0, 0)$: $4 - 2 = 2$, $8 - 2 = 6$, $4 + 2 = 6$. $\mathbf{v} = (-2, 1, 1, 0)$: $-2 + 1 + 1 = 0$, $-4 + 1 + 3 = 0$, $-2 - 1 + 3 = 0$. $\mathbf{w} = (1, -3, 0, 1)$: $1 - 3 + 2 = 0$, $2 - 3 + 1 = 0$, $1 + 3 - 4 = 0$. So the solution set is the particular solution $\mathbf{p}$ plus all solutions of the homogeneous system (Lecture 10).

(d) We need $4 - 2s + t = 0$ and $-2 + s - 3t = 0$. The first gives $t = 2s - 4$; then the second gives $-2 + s - 6s + 12 = 0$, so $s = 2$ and $t = 0$. The solution is $(0, 0, 2, 0)$. Check: $2 = 2$, $6 = 6$, $6 = 6$.

Problem 6. ► Recitation

Consider the system, where h and k are real numbers:

$$x + y + z = 1, \quad x + 2y + 3z = 2, \quad 2x + 3y + hz = k.$$

- For which values of h and k does the system have no solution, exactly one solution, or infinitely many solutions?
- Solve the system for $h = 5, k = 4$, and for $h = 4, k = 3$.

Solution. (a) Row reduce, keeping h and k as they are:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & h & k \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & h-2 & k-2 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & h-4 & k-3 \end{array} \right].$$

- If $h \neq 4$, the entry $h - 4$ is a pivot. Every column of the coefficient matrix has a pivot, so there is no free variable, and the last column has no pivot, so the system is consistent: **exactly one solution**.
- If $h = 4$ and $k \neq 3$, the last row says $0 = k - 3 \neq 0$: **no solution**.
- If $h = 4$ and $k = 3$, the last row says $0 = 0$, and z is free: **infinitely many solutions**.

(b) For $h = 5, k = 4$ the last row is $z = 1$. Then $y = 1 - 2z = -1$, and $x = 1 - y - z = 1 + 1 - 1 = 1$. So $(x, y, z) = (1, -1, 1)$. Check: $1 - 1 + 1 = 1$, $1 - 2 + 3 = 2$, $2 - 3 + 5 = 4$.

For $h = 4, k = 3$, the RREF is (after $R_1 \rightarrow R_1 - R_2$) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$, so with $z = t$ free,

$$\mathbf{x} = \begin{bmatrix} t \\ 1 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Check in the third equation: $2t + 3(1 - 2t) + 4t = 3$. In the row picture the three planes share this whole line.

Problem 7. ► Recitation

Let $\mathbf{v}_1 = (1, -1, 2)$, $\mathbf{v}_2 = (2, 1, 1)$ and $\mathbf{v}_3 = (1, 5, h)$ in $\mathbb{R}^3$.

- For which h is the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly dependent?
- For that h , find a dependence relation, and write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
- For that h , all three vectors lie in one plane through the origin. Find its equation.

Solution. (a) The list is dependent exactly when $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution, that is when the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ has a free variable:

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 5 \\ 2 & 1 & h \end{bmatrix} \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \\ 0 & -3 & h-2 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & h+4 \end{bmatrix}.$$

If $h \neq -4$ there are three pivots, no free variable, and only the trivial solution: independent. If $h = -4$ the third column has no pivot, c_3 is free, and the list is **dependent**.

(b) With $h = -4$, continue to the RREF: $R_2 \rightarrow \frac{1}{3}R_2$ gives $(0, 1, 2)$, then $R_1 \rightarrow R_1 - 2R_2$ gives $(1, 0, -3)$. So $c_1 = 3c_3$ and $c_2 = -2c_3$. Taking $c_3 = 1$:

$$3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

Check: $3(1, -1, 2) - 2(2, 1, 1) + (1, 5, -4) = (3-4+1, -3-2+5, 6-2-4) = (0, 0, 0)$. Hence $\mathbf{v}_3 = -3\mathbf{v}_1 + 2\mathbf{v}_2$.

(c) A normal vector is $\mathbf{v}_1 \times \mathbf{v}_2$ (Lecture 5) $= ((-1) \cdot 1 - 2 \cdot 1, 2 \cdot 2 - 1 \cdot 1, 1 \cdot 1 - (-1) \cdot 2) = (-3, 3, 3)$, or more simply $(1, -1, -1)$. The plane through the origin is

$$x - y - z = 0.$$

Check: $\mathbf{v}_1: 1 + 1 - 2 = 0$; $\mathbf{v}_2: 2 - 1 - 1 = 0$; $\mathbf{v}_3: 1 - 5 + 4 = 0$. Three vectors in one plane through the origin span at most that plane, which is why they are dependent.

Problem 8. ► Recitation

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 1 & 3 & 1 \end{bmatrix}$.

- For which vectors $\mathbf{b} = (b_1, b_2, b_3)$ does $A\mathbf{x} = \mathbf{b}$ have a solution? Describe this set geometrically.
- Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (1, 3, 2)$. Does it have a solution for $\mathbf{b} = (1, 0, 0)$?
- Are the columns of A linearly independent? If not, give a dependence relation.

Solution. (a) Row reduce with a general right-hand side:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 2 & 5 & 1 & b_2 \\ 1 & 3 & 1 & b_3 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - b_1 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_1 - b_2 + b_3 \end{array} \right].$$

(In the last entry, $(b_3 - b_1) - (b_2 - 2b_1) = b_1 - b_2 + b_3$.) The system is consistent exactly when

$$b_1 - b_2 + b_3 = 0.$$

Since $A\mathbf{x}$ is a combination of the columns, these $\mathbf{b}$ form the span of the columns: a plane through the origin with normal vector $(1, -1, 1)$. Each column lies on it: $1 - 2 + 1 = 0$, $2 - 5 + 3 = 0$, $0 - 1 + 1 = 0$.

(b) For $\mathbf{b} = (1, 3, 2)$: $1 - 3 + 2 = 0$, so there are solutions. The echelon form above has right-hand side

$(1, 3 - 2, 0) = (1, 1, 0)$; then $R_1 \rightarrow R_1 - 2R_2$ gives the RREF $\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$. With $x_3 = t$ free:

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Check with $t = 0$: $-1 \cdot (1, 2, 1) + 1 \cdot (2, 5, 3) = (1, 3, 2)$. For $\mathbf{b} = (1, 0, 0)$: $1 - 0 + 0 = 1 \neq 0$, so there is no solution.

(c) No. Any solution of $A\mathbf{x} = \mathbf{0}$ gives a relation among the columns, and the direction vector $(2, -1, 1)$ from (b) is one, since it solves $A\mathbf{x} = \mathbf{0}$ (Lecture 10): $2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 = (2-2+0, 4-5+1, 2-3+1) = (0, 0, 0)$.

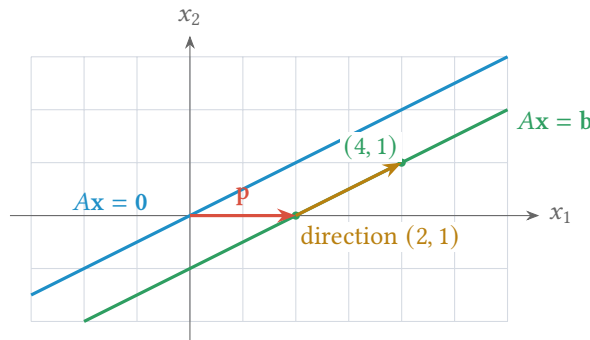
Problem 9. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$ and $\mathbf{b} = (2, -6)$.

- Solve $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$, each in parametric vector form.
- Draw both solution sets in the plane, with the particular solution and the direction vector.
- Ali finds the particular solution $(4, 1)$ instead. Show that his description gives the same solution set.
- Does $A\mathbf{x} = (1, 1)$ have a solution? Explain with the columns of A .

Solution. (a) $R_2 \rightarrow R_2 + 3R_1$ turns the second row into zeros in both systems (for $\mathbf{b}$: $-6 + 3 \cdot 2 = 0$). So $A\mathbf{x} = \mathbf{0}$ is the single equation $x_1 - 2x_2 = 0$, and $A\mathbf{x} = \mathbf{b}$ is the single equation $x_1 - 2x_2 = 2$; in both, $x_2 = t$ is free:

$$A\mathbf{x} = \mathbf{0} : \mathbf{x} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A\mathbf{x} = \mathbf{b} : \mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

- Both are lines with direction $(2, 1)$: the first through the origin, the second shifted by $\mathbf{p} = (2, 0)$.



Blue: the solutions of $A\mathbf{x} = \mathbf{0}$. Green: the solutions of $A\mathbf{x} = \mathbf{b}$, the blue line moved by $\mathbf{p}$. Red: the particular solution $\mathbf{p} = (2, 0)$, drawn from the origin. Gold: the direction vector $(2, 1)$, drawn from the tip of $\mathbf{p}$; it ends at Ali's particular solution $(4, 1)$.

- His description is $(4, 1) + u(2, 1)$. Since $(4, 1) = (2, 0) + 1 \cdot (2, 1)$, the point $(4, 1) + u(2, 1)$ equals $(2, 0) + (u+1)(2, 1)$, which is in our set; and our point $(2, 0) + t(2, 1)$ equals $(4, 1) + (t-1)(2, 1)$, which is in his. So the two sets are equal: any particular solution will do.

- $A\mathbf{x} = x_1(1, -3) + x_2(-2, 6) = (x_1 - 2x_2)(1, -3)$, so every $A\mathbf{x}$ is a multiple of $(1, -3)$. The vector $(1, 1)$ is not (its first entry would force the multiple 1, but $-3 \neq 1$), so there is no solution.

Problem 10. Consider the system

$$x + y = 0, \quad 2x + y + 2z = 1, \quad x + 3y + z = 3.$$

- Solve it over $\mathbb{R}$.
- Solve it over $\mathbb{Z}_5$ by bringing it to RREF. How many solutions are there? List them.
- Explain why the number of solutions differs.

Solution. Over any field, the first steps are $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 3 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 2 & 1 & 3 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 5 & 5 \end{array} \right].$$

(a) Over $\mathbb{R}$, $5z = 5$ gives $z = 1$; then $-y + 2 = 1$ gives $y = 1$, and $x = -y = -1$. One solution, $(-1, 1, 1)$. Check: $-1 + 1 = 0$, $-2 + 1 + 2 = 1$, $-1 + 3 + 1 = 3$.

(b) Over $\mathbb{Z}_5$, $5 = 0$, so the last row is $0 = 0$. In $\mathbb{Z}_5$, $-1 = 4$, $-2 = 3$ and $-3 = 2$. Scale R_2 by $(-1)^{-1} = -1$: the row $(0, -1, 2 | 1)$ becomes $(0, 1, -2 | -1) = (0, 1, 3 | 4)$. Then $R_1 \rightarrow R_1 - R_2$ gives $(1, 0, -3 | -4) = (1, 0, 2 | 1)$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

So $z = t$ is free, $x = 1 - 2t = 1 + 3t$ and $y = 4 - 3t = 4 + 2t$. The free variable takes the five values $t = 0, 1, 2, 3, 4$, giving exactly five solutions:

$$(1, 4, 0), \quad (4, 1, 1), \quad (2, 3, 2), \quad (0, 0, 3), \quad (3, 2, 4).$$

Check $(0, 0, 3)$: $0 = 0$, $2 \cdot 3 = 6 = 1$, $3 = 3$. (The real solution $(-1, 1, 1)$ reduces to $(4, 1, 1)$, the case $t = 1$.)

(c) The row operations are the same in both fields, but the third pivot candidate is 5, which is nonzero in $\mathbb{R}$ and zero in $\mathbb{Z}_5$. Over $\mathbb{Z}_5$ the third column loses its pivot, z becomes free, and a free variable over a field with 5 elements gives 5 solutions, not infinitely many.

C. Proof and concept

Problem 11. ► Recitation

True or false? Prove each true statement and give a counterexample for each false one. All systems are over $\mathbb{R}$.

- (a) A system with more equations than unknowns has no solution.
- (b) If a homogeneous system has a nontrivial solution, then it has infinitely many solutions.
- (c) If the list $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is linearly dependent, then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
- (d) If the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

Solution. (a) False. The system $x = 1$, $y = 2$, $x + y = 3$ has three equations in two unknowns, and $(1, 2)$ solves all three.

(b) True. Write the system as $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{h} \neq \mathbf{0}$ be a nontrivial solution. For every real t , linearity of the product (Lecture 12) gives

$$A(t\mathbf{h}) = t(A\mathbf{h}) = t\mathbf{0} = \mathbf{0},$$

so $t\mathbf{h}$ is a solution too. Different values of t give different vectors: if $t\mathbf{h} = r\mathbf{h}$ then $(t - r)\mathbf{h} = \mathbf{0}$, and since $\mathbf{h} \neq \mathbf{0}$ this forces $t - r = 0$ (Lecture 2). So there are infinitely many solutions, one for each real t . $\square$

(c) False. Take $\mathbf{u} = (1, 2)$, $\mathbf{v} = (-2, -4)$, $\mathbf{w} = (1, 0)$. The list is dependent, since $2\mathbf{u} + \mathbf{v} + 0\mathbf{w} = \mathbf{0}$. But every combination $a\mathbf{u} + b\mathbf{v} = (a - 2b)(1, 2)$ has second entry twice its first, and $\mathbf{w} = (1, 0)$ does not, so $\mathbf{w}$ is not a combination of $\mathbf{u}$ and $\mathbf{v}$. (A dependent list has *some* vector that is a combination of the others, here $\mathbf{v} = -2\mathbf{u}$, but not necessarily the last one.)

(d) False. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, whose columns $(1, 0, 0)$ and $(0, 1, 0)$ are independent (neither is a multiple of the other). Every $A\mathbf{x} = (x_1, x_2, 0)$ has third entry 0, so $A\mathbf{x} = (0, 0, 1)$ has no solution. Independent

columns mean a pivot in every *column*, which makes solutions unique when they exist; a solution for every $\mathbf{b}$ needs a pivot in every *row* (Lecture 12), and this A has three rows but only two pivots.

Problem 12. ► Recitation

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{F}^n$.

- (a) Over $\mathbb{F} = \mathbb{R}$: prove that if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, then so are $\mathbf{u} + \mathbf{v}$, $\mathbf{v} + \mathbf{w}$, $\mathbf{u} + \mathbf{w}$.
- (b) Show that the statement in (a) is false over $\mathbb{F} = \mathbb{Z}_2$. Which step of your proof breaks?
- (c) Prove that $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, $\mathbf{w} - \mathbf{u}$ are linearly dependent, over any field and for any $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Solution. (a) Suppose $a(\mathbf{u} + \mathbf{v}) + b(\mathbf{v} + \mathbf{w}) + c(\mathbf{u} + \mathbf{w}) = \mathbf{0}$ for real numbers a, b, c . We must show $a = b = c = 0$. Expanding with the distributive law $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$, then regrouping with the commutativity and associativity of vector addition, and collecting with the other distributive law $(a + c)\mathbf{u} = a\mathbf{u} + c\mathbf{u}$:

$$(a + c)\mathbf{u} + (a + b)\mathbf{v} + (b + c)\mathbf{w} = \mathbf{0}.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are independent, all three weights are 0:

$$a + c = 0, \quad a + b = 0, \quad b + c = 0.$$

From the second and third, $a = -b = c$. Then the first gives $2a = 0$, and multiplying by $\frac{1}{2}$ gives $a = 0$. Hence $c = a = 0$ and $b = -a = 0$. $\square$

(b) In $\mathbb{Z}_2$, $1 + 1 = 0$. Take any independent $\mathbf{u}, \mathbf{v}, \mathbf{w}$ over $\mathbb{Z}_2$, for example the standard vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ of $\mathbb{Z}_2^3$. Then

$$1(\mathbf{u} + \mathbf{v}) + 1(\mathbf{v} + \mathbf{w}) + 1(\mathbf{u} + \mathbf{w}) = (1 + 1)\mathbf{u} + (1 + 1)\mathbf{v} + (1 + 1)\mathbf{w} = 0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w} = \mathbf{0},$$

a relation with nonzero weights, so the three sums are dependent. (Concretely: $(1, 1, 0) + (0, 1, 1) + (1, 0, 1) = (1 + 1, 1 + 1, 1 + 1) = (0, 0, 0)$ in $\mathbb{Z}_2^3$, the example of Lecture 11.) The proof breaks at “ $2a = 0$ gives $a = 0$ ”: it multiplied by $\frac{1}{2}$, the inverse of 2, and in $\mathbb{Z}_2$ we have $2 = 1 + 1 = 0$, and 0 has no inverse. Over any field in which $2 \neq 0$ (such as $\mathbb{Q}, \mathbb{C}$ or $\mathbb{Z}_p$ for an odd prime p), the proof of (a) works unchanged.

(c) Adding the three vectors, each of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ appears once with a plus sign and once with a minus sign:

$$1(\mathbf{u} - \mathbf{v}) + 1(\mathbf{v} - \mathbf{w}) + 1(\mathbf{w} - \mathbf{u}) = \mathbf{0}.$$

The weights 1, 1, 1 are not all zero (in any field $1 \neq 0$), so the list is dependent. $\square$

Problem 13.

- (a) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent vectors in $\mathbb{F}^n$, over any field $\mathbb{F}$. Prove that $\mathbf{v}_1$, $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent. Compare with Problem 12: why does no field cause trouble here?
- (b) Let A be an $m \times n$ matrix with linearly independent columns, and let B be obtained from A by an elementary row operation. Prove that the columns of B are linearly independent.
- (c) Give three vectors in $\mathbb{R}^3$, none of them a multiple of another, that are linearly dependent. Why does this not contradict the fact (Lecture 11) that two vectors are dependent exactly when one is a multiple of the other?

Solution. (a) Suppose $a\mathbf{v}_1 + b(\mathbf{v}_1 + \mathbf{v}_2) + c(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}$. Expanding with the distributive law $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$, regrouping with the commutativity and associativity of vector addition, and collecting with $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$:

$$(a + b + c)\mathbf{v}_1 + (b + c)\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}.$$

By independence, $c = 0$, $b + c = 0$ and $a + b + c = 0$. The first gives $c = 0$, then the second gives $b = 0$, then the third gives $a = 0$. $\square$

Here the system for a, b, c , read with the unknowns in the order c, b, a , is already a staircase with pivots 1, 1, 1, so it is solved by back substitution without dividing by anything. In Problem 12 the system had to be divided by 2, which is impossible in $\mathbb{Z}_2$. Since $1 \neq 0$ in every field, this proof works in every field.

(b) The columns of A are independent exactly when $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$, since $A\mathbf{x}$ is the combination of the columns with weights $x_1, \dots, x_n$ (Lecture 12). The homogeneous system with augmented matrix $[A \mid \mathbf{0}]$ becomes $[B \mid \mathbf{0}]$ under the row operation (a column of zeros stays zero). Row operations do not change the solution set (Lecture 8), so $B\mathbf{x} = \mathbf{0}$ also has only the solution $\mathbf{x} = \mathbf{0}$, and the columns of B are independent. $\square$

(c) For example $(1, 2, 0)$, $(0, 1, 1)$, $(1, 3, 1)$: no one is a multiple of another (compare the zero entries), but $(1, 2, 0) + (0, 1, 1) - (1, 3, 1) = (0, 0, 0)$. The fact about two vectors is the case $k = 2$ of the theorem that a list is dependent exactly when one vector is a combination of the *others*. With three vectors, that combination may use both of the other two, as here: $(1, 3, 1)$ is the sum of the other two, not a multiple of either.

D. Challenge and application

Problem 14. Balancing a chemical equation means finding whole numbers of molecules so that each element has as many atoms on the left as on the right.

- (a) Ethanol burns in oxygen: $x_1 \text{C}_2\text{H}_6\text{O} + x_2 \text{O}_2 \rightarrow x_3 \text{CO}_2 + x_4 \text{H}_2\text{O}$. Write one equation per element (C, H, O), solve the homogeneous system by bringing it to RREF, and give the balanced equation with the smallest positive whole numbers.
- (b) Do the same for photosynthesis: $x_1 \text{CO}_2 + x_2 \text{H}_2\text{O} \rightarrow x_3 \text{C}_6\text{H}_{12}\text{O}_6 + x_4 \text{O}_2$.
- (c) Explain, without solving, why a reaction with four kinds of molecules made of three elements always has a solution $\mathbf{x} \neq \mathbf{0}$ of the balancing equations (possibly with negative or fractional entries).

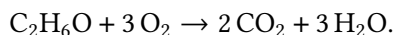
Solution. (a) Carbon: $2x_1 = x_3$. Hydrogen: $6x_1 = 2x_4$. Oxygen: $x_1 + 2x_2 = 2x_3 + x_4$. Moving everything to the left:

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 6 & 0 & 0 & -2 \\ 1 & 2 & -2 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 6 & 0 & 0 & -2 \\ 2 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{\begin{matrix} R_2 \rightarrow R_2 - 6R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{matrix}} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -12 & 12 & 4 \\ 0 & -4 & 3 & 2 \end{bmatrix}.$$

Scale $R_2 \rightarrow -\frac{1}{12}R_2$ to get $(0, 1, -1, -\frac{1}{3})$; then $R_3 \rightarrow R_3 + 4R_2$ gives $(0, 0, -1, \frac{2}{3})$, and $R_3 \rightarrow -R_3$ gives $(0, 0, 1, -\frac{2}{3})$. Clearing above the third pivot ($R_2 \rightarrow R_2 + R_3$, $R_1 \rightarrow R_1 + 2R_3$) and then above the second ($R_1 \rightarrow R_1 - 2R_2$):

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{2}{3} \end{bmatrix}.$$

So $x_4 = t$ is free, and $x_1 = \frac{1}{3}t$, $x_2 = t$, $x_3 = \frac{2}{3}t$. The smallest positive whole numbers come from $t = 3$: $(x_1, x_2, x_3, x_4) = (1, 3, 2, 3)$,

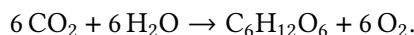


Check: C $2 = 2$; H $6 = 6$; O $1 + 6 = 4 + 3$.

(b) Carbon: $x_1 = 6x_3$. Hydrogen: $2x_2 = 12x_3$. Oxygen: $2x_1 + x_2 = 6x_3 + 2x_4$. In matrix form, and reduced:

$$\begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 2 & -12 & 0 \\ 2 & 1 & -6 & -2 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 2 & -12 & 0 \\ 0 & 1 & 6 & -2 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 1 & 6 & -2 \end{bmatrix} \\ \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 12 & -2 \end{bmatrix}.$$

Then $R_3 \rightarrow \frac{1}{12}R_3$ gives $(0, 0, 1, -\frac{1}{6})$, and clearing above it, $R_1 \rightarrow R_1 + 6R_3$ and $R_2 \rightarrow R_2 + 6R_3$, gives the RREF with rows $(1, 0, 0, -1)$, $(0, 1, 0, -1)$, $(0, 0, 1, -\frac{1}{6})$. So $x_1 = x_2 = t$, $x_3 = \frac{1}{6}t$, $x_4 = t$, and $t = 6$ gives $(6, 6, 1, 6)$:



Check: C $6 = 6$; H $12 = 12$; O $12 + 6 = 6 + 12$.

(c) The balancing conditions form a homogeneous system of 3 equations (one per element) in 4 unknowns. A homogeneous system with more unknowns than equations always has a nontrivial solution (Lecture 10). (Whether one of them has all entries positive is a question about the chemistry, not about linear algebra; if all coefficients are rational, as here, a positive rational solution can be scaled to whole numbers.)

Problem 15. The RREF of a matrix is unique (Lecture 9), so counting the possible RREFs counts the essentially different ways a system can look after elimination. Work over the field $\mathbb{Z}_2 = \{0, 1\}$.

- List all 2×2 matrices over $\mathbb{Z}_2$ that are in RREF. How many are there?
- How many 2×3 matrices over $\mathbb{Z}_2$ are in RREF? Sort them by their pivot columns.
- For each 2×3 RREF matrix R in (b), how many solutions does $R\mathbf{x} = \mathbf{0}$ have in $\mathbb{Z}_2^3$?
- How many 2×2 matrices over $\mathbb{Z}_p$, p prime, are in RREF?

Solution. In RREF every pivot is 1, every other entry of a pivot column is 0, and the only free choices are the entries in non-pivot columns to the right of a pivot, in the pivot rows.

(a) Sorted by pivot columns:

$$\text{none: } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad \text{column 1: } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; \quad \text{column 2: } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \text{columns 1, 2: } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are 5.

(b) Write * for a free entry (0 or 1).

- No pivot: the zero matrix, 1.
- One pivot, in column 1: first row $(1, *, *)$, 4 matrices; in column 2: $(0, 1, *)$, 2; in column 3: $(0, 0, 1)$, 1. Total 7.
- Two pivots, in columns 1, 2: rows $(1, 0, *)$, $(0, 1, *)$, 4 matrices; in columns 1, 3: rows $(1, *, 0)$, $(0, 0, 1)$, 2; in columns 2, 3: rows $(0, 1, 0)$, $(0, 0, 1)$, 1. Total 7.

Altogether $1 + 7 + 7 = 15$.

(c) The number of free variables is 3 minus the number of pivots, and each free variable takes 2 values in $\mathbb{Z}_2$, independently: the pivot variables are then fixed. So: the zero matrix has $2^3 = 8$ solutions; each of the 7 matrices with one pivot has $2^2 = 4$; each of the 7 with two pivots has $2^1 = 2$.

(d) As in (a), with the free entry now taking p values: $1 + p + 1 + 1 = p + 3$. For $p = 2$ this is 5, as in (a).