

Problem Sheet 3

Lectures 9 to 12: reduced row echelon form; solution sets; linear independence; the matrix-vector product. Problems marked ► are solved in Recitation 3; the others are for practice on your own, with full solutions in the solutions booklet.

A. Warm-up

Problem 1. Recall (Lecture 9): a matrix is in *reduced row echelon form* (RREF) when it is in row echelon form, every pivot is 1, and every other entry in a pivot column is 0. For each matrix, say whether it is in RREF. If it is not, name a condition that fails. If it is, name its pivot columns.

$$\begin{array}{lll} \text{(a)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} & \text{(b)} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \text{(c)} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \\ \text{(d)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \text{(e)} \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{bmatrix} & \end{array}$$

Then bring (c) and (e) to RREF.

Problem 2. Recall (Lecture 12): if A has columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ and $\mathbf{x} = (x_1, \dots, x_n)$, then $A\mathbf{x} = x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n$.

- Compute $A\mathbf{x}$ for $A = \begin{bmatrix} 3 & 0 & -2 \\ 1 & 4 & 1 \end{bmatrix}$ and $\mathbf{x} = (1, 2, -1)$, first as a combination of the columns, then by the row rule (entry i of $A\mathbf{x}$ is row i of A times $\mathbf{x}$, entry by entry and added up). Check that the two answers agree.
- Explain why $A\mathbf{y}$ is not defined for $\mathbf{y} = (1, 2)$.
- Let $B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{x} = (1, 2)$. Compute $B\mathbf{x}$, and draw the column picture: the columns $\mathbf{b}_1, \mathbf{b}_2$ of B , the combination $1\mathbf{b}_1 + 2\mathbf{b}_2$ tip to tail, and the result.

Problem 3.

- Write the system $3x_1 - x_2 + 2x_3 = 4$, $x_1 + 5x_3 = -1$ as a vector equation (a combination of the columns equal to the right-hand side) and as a matrix equation $A\mathbf{x} = \mathbf{b}$.
- Write the vector equation $x_1(1, 2, 0) + x_2(-1, 0, 3) = (4, 1, 1)$ as a system of equations and as $A\mathbf{x} = \mathbf{b}$.
- If A is a 4×3 matrix and $A\mathbf{x} = \mathbf{b}$, how many entries do $\mathbf{x}$ and $\mathbf{b}$ have?

Problem 4. Decide by inspection whether each list of vectors is linearly independent, with a one-line reason.

- $(1, -2), (-3, 6)$ in $\mathbb{R}^2$.
- $(1, 0, 2), (0, 0, 0), (3, 1, 1)$ in $\mathbb{R}^3$.
- $(2, 1), (1, 3), (4, -3)$ in $\mathbb{R}^2$. (Also find a dependence relation.)
- $(1, 1, 0), (0, 1, 1)$ in $\mathbb{R}^3$.

B. Core

Problem 5. ► Recitation

Consider the system

$$x_1 + x_2 + x_3 + 2x_4 = 2, \quad 2x_1 + x_2 + 3x_3 + x_4 = 6, \quad x_1 - x_2 + 3x_3 - 4x_4 = 6.$$

- (a) Row reduce the augmented matrix to RREF. Which are the pivot columns, and which variables are free?
- (b) Write the general solution in parametric vector form, $\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}$.
- (c) Check that $\mathbf{p}$ solves the system and that $\mathbf{v}$ and $\mathbf{w}$ solve the homogeneous system with the same coefficients.
- (d) Find the solution with $x_1 = 0$ and $x_2 = 0$.

Problem 6. ► Recitation

Consider the system, where h and k are real numbers:

$$x + y + z = 1, \quad x + 2y + 3z = 2, \quad 2x + 3y + hz = k.$$

- (a) For which values of h and k does the system have no solution, exactly one solution, or infinitely many solutions?
- (b) Solve the system for $h = 5, k = 4$, and for $h = 4, k = 3$.

Problem 7. ► Recitation

Let $\mathbf{v}_1 = (1, -1, 2)$, $\mathbf{v}_2 = (2, 1, 1)$ and $\mathbf{v}_3 = (1, 5, h)$ in $\mathbb{R}^3$.

- (a) For which h is the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly dependent?
- (b) For that h , find a dependence relation, and write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
- (c) For that h , all three vectors lie in one plane through the origin. Find its equation.

Problem 8. ► Recitation

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$

- (a) For which vectors $\mathbf{b} = (b_1, b_2, b_3)$ does $A\mathbf{x} = \mathbf{b}$ have a solution? Describe this set geometrically.
- (b) Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (1, 3, 2)$. Does it have a solution for $\mathbf{b} = (1, 0, 0)$?
- (c) Are the columns of A linearly independent? If not, give a dependence relation.

Problem 9. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$ and $\mathbf{b} = (2, -6)$.

- (a) Solve $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$, each in parametric vector form.

- (b) Draw both solution sets in the plane, with the particular solution and the direction vector.
- (c) Ali finds the particular solution $(4, 1)$ instead. Show that his description gives the same solution set.
- (d) Does $Ax = (1, 1)$ have a solution? Explain with the columns of A .

Problem 10. Consider the system

$$x + y = 0, \quad 2x + y + 2z = 1, \quad x + 3y + z = 3.$$

- (a) Solve it over $\mathbb{R}$.
- (b) Solve it over $\mathbb{Z}_5$ by bringing it to RREF. How many solutions are there? List them.
- (c) Explain why the number of solutions differs.

C. Proof and concept

Problem 11. ► Recitation

True or false? Prove each true statement and give a counterexample for each false one. All systems are over $\mathbb{R}$.

- (a) A system with more equations than unknowns has no solution.
- (b) If a homogeneous system has a nontrivial solution, then it has infinitely many solutions.
- (c) If the list $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is linearly dependent, then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
- (d) If the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

Problem 12. ► Recitation

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{F}^n$.

- (a) Over $\mathbb{F} = \mathbb{R}$: prove that if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, then so are $\mathbf{u} + \mathbf{v}$, $\mathbf{v} + \mathbf{w}$, $\mathbf{u} + \mathbf{w}$.
- (b) Show that the statement in (a) is false over $\mathbb{F} = \mathbb{Z}_2$. Which step of your proof breaks?
- (c) Prove that $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, $\mathbf{w} - \mathbf{u}$ are linearly dependent, over any field and for any $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Problem 13.

- (a) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent vectors in $\mathbb{F}^n$, over any field $\mathbb{F}$. Prove that $\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent. Compare with Problem 12: why does no field cause trouble here?
- (b) Let A be an $m \times n$ matrix with linearly independent columns, and let B be obtained from A by an elementary row operation. Prove that the columns of B are linearly independent.
- (c) Give three vectors in $\mathbb{R}^3$, none of them a multiple of another, that are linearly dependent. Why does this not contradict the fact (Lecture 11) that two vectors are dependent exactly when one is a multiple of the other?

D. Challenge and application

Problem 14. Balancing a chemical equation means finding whole numbers of molecules so that each element has as many atoms on the left as on the right.

- (a) Ethanol burns in oxygen: $x_1 \text{C}_2\text{H}_6\text{O} + x_2 \text{O}_2 \rightarrow x_3 \text{CO}_2 + x_4 \text{H}_2\text{O}$. Write one equation per element (C, H, O), solve the homogeneous system by bringing it to RREF, and give the balanced equation with the smallest positive whole numbers.
- (b) Do the same for photosynthesis: $x_1 \text{CO}_2 + x_2 \text{H}_2\text{O} \rightarrow x_3 \text{C}_6\text{H}_{12}\text{O}_6 + x_4 \text{O}_2$.
- (c) Explain, without solving, why a reaction with four kinds of molecules made of three elements always has a solution $\mathbf{x} \neq \mathbf{0}$ of the balancing equations (possibly with negative or fractional entries).

Problem 15. The RREF of a matrix is unique (Lecture 9), so counting the possible RREFs counts the essentially different ways a system can look after elimination. Work over the field $\mathbb{Z}_2 = \{0, 1\}$.

- (a) List all 2×2 matrices over $\mathbb{Z}_2$ that are in RREF. How many are there?
- (b) How many 2×3 matrices over $\mathbb{Z}_2$ are in RREF? Sort them by their pivot columns.
- (c) For each 2×3 RREF matrix R in (b), how many solutions does $R\mathbf{x} = \mathbf{0}$ have in $\mathbb{Z}_2^3$?
- (d) How many 2×2 matrices over $\mathbb{Z}_p$, p prime, are in RREF?