

Problem Sheet 4: Solutions

Lectures 13 to 16: linear transformations of $\mathbb{R}^n$; matrix multiplication; matrix algebra; inverse matrices. Problems marked ► are solved in Recitation 4; the others are for practice on your own, with full solutions in the solutions booklet.

A. Warm-up

Problem 1. Recall (Lecture 13): a map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *linear* when $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{u})$ for all vectors $\mathbf{u}, \mathbf{v}$ and all scalars c . Decide whether each map is linear. For each linear map, give its standard matrix; for each map that is not linear, show one rule that fails, with numbers.

(a) $T(x, y) = (2x - y, 3y)$

(b) $T(x, y) = (x + y, 2)$

(c) $T(x, y) = (x, |y|)$

(d) $T(x, y, z) = (x - z, 0)$

Solution. (a) Linear: each output entry is a fixed combination of x and y , so $T(x, y) = A\mathbf{x}$ with $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$. Its columns are $T(\mathbf{e}_1) = T(1, 0) = (2, 0)$ and $T(\mathbf{e}_2) = T(0, 1) = (-1, 3)$.

(b) Not linear. With $\mathbf{u} = (0, 0)$ and $c = 2$: $T(2\mathbf{u}) = T(0, 0) = (0, 2)$, but $2T(\mathbf{u}) = (0, 4)$, so the scaling rule fails. (Equivalently, a linear map sends $\mathbf{0}$ to $\mathbf{0}$, Lecture 13.)

(c) Not linear, although $T(\mathbf{0}) = \mathbf{0}$. With $\mathbf{u} = (0, 1)$ and $c = -1$: $T(-\mathbf{u}) = T(0, -1) = (0, 1)$, but $-T(\mathbf{u}) = -(0, 1) = (0, -1)$.

(d) Linear, from $\mathbb{R}^3$ to $\mathbb{R}^2$, with the 2×3 standard matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, whose columns are $T(\mathbf{e}_1) = (1, 0)$, $T(\mathbf{e}_2) = (0, 0)$ and $T(\mathbf{e}_3) = (-1, 0)$.

Problem 2. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$.

(a) Compute AB and BA . Do A and B commute?

(b) Compute AC . What is its size?

(c) Explain why CA is not defined.

Solution. Entry (i, j) of a product is row i of the left factor times column j of the right factor (Lecture 14).

(a) $AB = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 1 & 1 \cdot 1 + 2 \cdot 0 \\ 0 \cdot 3 + (-1) \cdot 1 & 0 \cdot 1 + (-1) \cdot 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 3 \cdot 1 + 1 \cdot 0 & 3 \cdot 2 + 1 \cdot (-1) \\ 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 2 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. They do not commute: $AB \neq BA$.

(b) A is 2×2 and C is 2×3 , so AC is 2×3 :

$$AC = \begin{bmatrix} 1 \cdot 1 + 2 \cdot (-1) & 1 \cdot 0 + 2 \cdot 1 & 1 \cdot 2 + 2 \cdot 0 \\ 0 \cdot 1 + (-1) \cdot (-1) & 0 \cdot 0 + (-1) \cdot 1 & 0 \cdot 2 + (-1) \cdot 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

(c) CA needs the number of columns of C (3) to equal the number of rows of A (2). A row of C has three entries and a column of A has two, so they cannot be multiplied entry by entry.

Problem 3. Let $M = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 5 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 3 & -1 \\ 6 & 1 & 2 \end{bmatrix}$.

- (a) Compute MN and $(MN)^T$. Then compute $N^T M^T$ and $M^T N^T$. Which of the two equals $(MN)^T$?
 (b) Write A as the sum of a symmetric matrix S and a skew-symmetric matrix K (one with $K^T = -K$).

Solution. (a) M is 2×3 and N is 3×2 , so MN is 2×2 . Row times column:

$$MN = \begin{bmatrix} 2 - 2 + 0 & 0 - 1 + 0 \\ 4 + 6 + 0 & 0 + 3 + 15 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 10 & 18 \end{bmatrix}, \quad (MN)^T = \begin{bmatrix} 0 & 10 \\ -1 & 18 \end{bmatrix}.$$

With $N^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and $M^T = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ 0 & 5 \end{bmatrix}$:

$$N^T M^T = \begin{bmatrix} 2 - 2 + 0 & 4 + 6 + 0 \\ 0 - 1 + 0 & 0 + 3 + 15 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ -1 & 18 \end{bmatrix}, \quad M^T N^T = \begin{bmatrix} 2 & 8 & 12 \\ -1 & 1 & 9 \\ 0 & 5 & 15 \end{bmatrix}.$$

So $(MN)^T = N^T M^T$, with the order of the factors reversed (Lecture 15). The product $M^T N^T$ is not even the right size: it is 3×3 .

(b) Take $S = \frac{1}{2}(A + A^T)$ and $K = \frac{1}{2}(A - A^T)$. Then $S^T = \frac{1}{2}(A^T + A) = S$, $K^T = \frac{1}{2}(A^T - A) = -K$, and $S + K = A$. With $A^T = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix}$:

$$S = \frac{1}{2} \begin{bmatrix} 2 & 6 & 6 \\ 6 & 6 & 0 \\ 6 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 2 \end{bmatrix}, \quad K = \frac{1}{2} \begin{bmatrix} 0 & 2 & -6 \\ -2 & 0 & -2 \\ 6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}.$$

Check: $S + K = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 3 & -1 \\ 6 & 1 & 2 \end{bmatrix} = A$. The diagonal of K is zero, as it must be for a real skew-symmetric matrix.

Problem 4. Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$.

- (a) Find A^{-1} with the formula for 2×2 inverses (Lecture 16), and check that $AA^{-1} = I$.
 (b) Use A^{-1} to solve $A\mathbf{x} = (5, 1)$.
 (c) Show that B is not invertible.

Solution. (a) For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ the inverse is $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Here $ad - bc = 4 \cdot 1 - 3 \cdot 1 = 1$, so

$$A^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}, \quad AA^{-1} = \begin{bmatrix} 4 \cdot 1 + 3 \cdot (-1) & 4 \cdot (-3) + 3 \cdot 4 \\ 1 \cdot 1 + 1 \cdot (-1) & 1 \cdot (-3) + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) $\mathbf{x} = A^{-1}\mathbf{b} = (1 \cdot 5 - 3 \cdot 1, -1 \cdot 5 + 4 \cdot 1) = (2, -1)$. Check: $4 \cdot 2 + 3 \cdot (-1) = 5$ and $2 - 1 = 1$.

(c) Here $ad - bc = 2 \cdot 2 - 4 \cdot 1 = 0$. Directly: the second column of B is twice the first, so $B(2, -1) = 2(2, 1) - (4, 2) = \mathbf{0}$. If B had an inverse, then $(2, -1) = B^{-1}B(2, -1) = B^{-1}\mathbf{0} = \mathbf{0}$, which is false. So B is not invertible.

B. Core

Problem 5. ► Recitation

Let L be the line through the origin in the direction $\mathbf{u} = (2, 1)$. The *projection* onto L sends each $\mathbf{x}$ in $\mathbb{R}^2$ to the point of L closest to it,

$$P(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u},$$

and the *reflection* across L is $S(\mathbf{x}) = 2P(\mathbf{x}) - \mathbf{x}$.

- Find the standard matrix of P by computing $P(\mathbf{e}_1)$ and $P(\mathbf{e}_2)$.
- Find the standard matrix of S .
- Compute $P(\mathbf{x})$ and $S(\mathbf{x})$ for $\mathbf{x} = (3, -1)$, and check that $S(\mathbf{x})$ has the same length as $\mathbf{x}$.
- Draw L , $\mathbf{x}$, $P(\mathbf{x})$ and $S(\mathbf{x})$.
- Explain geometrically why $P^2 = P$ and $S^2 = I$, and check $S^2 = I$ by multiplying the matrices.

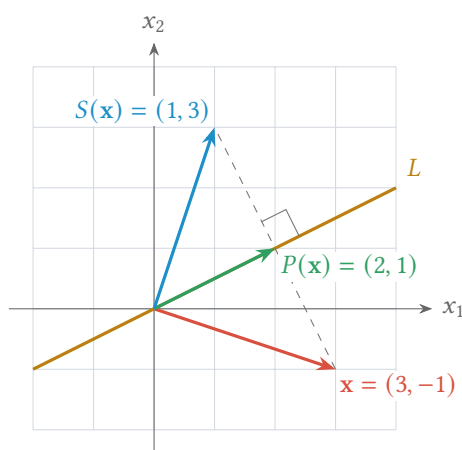
Solution. (a) $\mathbf{u} \cdot \mathbf{u} = 4 + 1 = 5$. Since $\mathbf{e}_1 \cdot \mathbf{u} = 2$ and $\mathbf{e}_2 \cdot \mathbf{u} = 1$,

$$P(\mathbf{e}_1) = \frac{2}{5} (2, 1) = \left(\frac{4}{5}, \frac{2}{5}\right), \quad P(\mathbf{e}_2) = \frac{1}{5} (2, 1) = \left(\frac{2}{5}, \frac{1}{5}\right), \quad P = \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$(b) S = 2P - I = \begin{bmatrix} \frac{8}{5} - 1 & \frac{4}{5} \\ \frac{4}{5} & \frac{2}{5} - 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

(c) $\mathbf{x} \cdot \mathbf{u} = 6 - 1 = 5$, so $P(\mathbf{x}) = \frac{5}{5} (2, 1) = (2, 1)$. With the matrix: $\frac{1}{5}(4 \cdot 3 + 2 \cdot (-1), 2 \cdot 3 + 1 \cdot (-1)) = \frac{1}{5}(10, 5) = (2, 1)$. Then $S(\mathbf{x}) = 2(2, 1) - (3, -1) = (1, 3)$; with the matrix, $\frac{1}{5}(9 - 4, 12 + 3) = (1, 3)$. Lengths: $\|\mathbf{x}\| = \sqrt{9 + 1} = \sqrt{10}$ and $\|S(\mathbf{x})\| = \sqrt{1 + 9} = \sqrt{10}$.

(d) $P(\mathbf{x})$ is the midpoint of $\mathbf{x}$ and $S(\mathbf{x})$, and the segment from $\mathbf{x}$ to $S(\mathbf{x})$ crosses L at a right angle.



Gold: the line L . Red: $\mathbf{x}$. Green: its projection $P(\mathbf{x})$. Blue: its reflection $S(\mathbf{x})$. Dashed: the segment from $\mathbf{x}$ to $S(\mathbf{x})$, perpendicular to L and cut in half by $P(\mathbf{x})$.

(e) $P(\mathbf{x})$ already lies on L , and the closest point of L to a point of L is the point itself, so projecting twice changes nothing: $P^2 = P$. Reflecting twice across the same line brings every point back where it started, so $S^2 = I$. By multiplying:

$$S^2 = \frac{1}{25} \begin{bmatrix} 3 \cdot 3 + 4 \cdot 4 & 3 \cdot 4 + 4 \cdot (-3) \\ 4 \cdot 3 + (-3) \cdot 4 & 4 \cdot 4 + (-3)(-3) \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = I.$$

(Similarly $P^2 = \frac{1}{25} \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} = P$.)

Problem 6. ► Recitation

A linear map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ satisfies $T(1, 1) = (1, 2, 0)$ and $T(1, -1) = (3, 0, 2)$.

- (a) Write $\mathbf{e}_1$ and $\mathbf{e}_2$ as combinations of $(1, 1)$ and $(1, -1)$, and find $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.
- (b) Find the standard matrix of T and compute $T(5, 1)$.
- (c) Check $T(5, 1)$ a second way, by writing $(5, 1)$ as a combination of $(1, 1)$ and $(1, -1)$.
- (d) Is there a linear map with $T(1, 1) = (1, 2, 0)$ and $T(2, 2) = (2, 5, 0)$? Explain.

Solution. (a) $\mathbf{e}_1 = \frac{1}{2}(1, 1) + \frac{1}{2}(1, -1)$ and $\mathbf{e}_2 = \frac{1}{2}(1, 1) - \frac{1}{2}(1, -1)$. By linearity,

$$T(\mathbf{e}_1) = \frac{1}{2}(1, 2, 0) + \frac{1}{2}(3, 0, 2) = (2, 1, 1), \quad T(\mathbf{e}_2) = \frac{1}{2}(1, 2, 0) - \frac{1}{2}(3, 0, 2) = (-1, 1, -1).$$

(b) The columns of the standard matrix are $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad T(5, 1) = A \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}.$$

(c) Solving $a(1, 1) + b(1, -1) = (5, 1)$: $a + b = 5$ and $a - b = 1$, so $a = 3$, $b = 2$. Then $T(5, 1) = 3(1, 2, 0) + 2(3, 0, 2) = (3 + 6, 6 + 0, 0 + 4) = (9, 6, 4)$, the same.

(d) No. Since $(2, 2) = 2(1, 1)$, linearity forces $T(2, 2) = 2T(1, 1) = (2, 4, 0)$, not $(2, 5, 0)$.

Problem 7. ► Recitation

Let R be the rotation of the plane by 90° counterclockwise and F the reflection across the x_1 -axis.

- (a) Write down the standard matrices of R and F (Lecture 13), from the images of $\mathbf{e}_1$ and $\mathbf{e}_2$.
- (b) Compute RF and FR . Which map does each product stand for (first which map, then which)? Do R and F commute?
- (c) Describe RF and FR geometrically, and draw $\mathbf{v} = (3, 1)$, $RF\mathbf{v}$ and $FR\mathbf{v}$.
- (d) Compute R^2 and R^4 , and explain the answers geometrically.

Solution. (a) R turns $\mathbf{e}_1$ into $\mathbf{e}_2 = (0, 1)$ and $\mathbf{e}_2$ into $-\mathbf{e}_1 = (-1, 0)$; F keeps $\mathbf{e}_1$ and turns $\mathbf{e}_2$ into $-\mathbf{e}_2$:

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

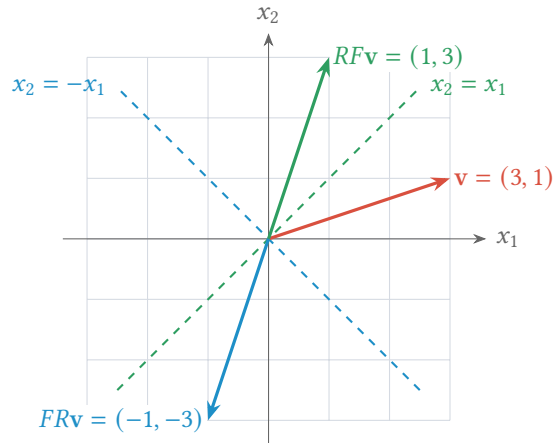
(b) $RF\mathbf{x} = R(F\mathbf{x})$, so RF is “first F , then R ”; and FR is “first R , then F ” (Lecture 14).

$$RF = \begin{bmatrix} 0 \cdot 1 + (-1) \cdot 0 & 0 \cdot 0 + (-1)(-1) \\ 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$FR = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 1 & 1 \cdot (-1) + 0 \cdot 0 \\ 0 \cdot 0 + (-1) \cdot 1 & 0 \cdot (-1) + (-1) \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

$RF \neq FR$, so R and F do not commute: the order of the two moves matters.

(c) RF swaps the two coordinates, $(x_1, x_2) \mapsto (x_2, x_1)$: it is the reflection across the line $x_2 = x_1$. FR sends $(x_1, x_2) \mapsto (-x_2, -x_1)$: the reflection across the line $x_2 = -x_1$. For $\mathbf{v} = (3, 1)$: $RF\mathbf{v} = (1, 3)$ and $FR\mathbf{v} = (-1, -3)$. Step by step: $F\mathbf{v} = (3, -1)$, then $R(3, -1) = (1, 3)$; and $R\mathbf{v} = (-1, 3)$, then $F(-1, 3) = (-1, -3)$.



Red: $\mathbf{v}$. Green: $R\mathbf{F}\mathbf{v}$, the mirror image of $\mathbf{v}$ in the green dashed line $x_2 = x_1$. Blue: $F\mathbf{R}\mathbf{v}$, the mirror image of $\mathbf{v}$ in the blue dashed line $x_2 = -x_1$.

(d)

$$R^2 = \begin{bmatrix} 0 \cdot 0 + (-1) \cdot 1 & 0 \cdot (-1) + (-1) \cdot 0 \\ 1 \cdot 0 + 0 \cdot 1 & 1 \cdot (-1) + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I,$$

the rotation by 180° , which sends every $\mathbf{x}$ to $-\mathbf{x}$. Then $R^4 = R^2 R^2 = (-I)(-I) = I$: four quarter turns make a full turn, which moves nothing.

Problem 8. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

- (a) Compute $(A + B)^2$ and $A^2 + 2AB + B^2$. Are they equal?
- (b) Expand $(A + B)^2 = (A + B)(A + B)$ with the distributive laws, and check your expansion with these A and B . When is $(A + B)^2 = A^2 + 2AB + B^2$?
- (c) Find all 2×2 matrices X with $AX = XA$.

Solution. (a) $A + B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, so $(A + B)^2 = \begin{bmatrix} 1+2 & 2+2 \\ 1+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Next,

$$A^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix},$$

so $A^2 + 2AB + B^2 = \begin{bmatrix} 1+4+0 & 4+0+0 \\ 0+2+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & 1 \end{bmatrix}$. They are not equal.

(b) $(A + B)(A + B) = A(A + B) + B(A + B) = A^2 + AB + BA + B^2$. Here

$$A^2 + AB + BA + B^2 = \begin{bmatrix} 1+2+0+0 & 4+0+0+0 \\ 0+1+1+0 & 1+0+2+0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix},$$

as in (a). So $(A + B)^2 = A^2 + 2AB + B^2$ exactly when $AB + BA = 2AB$, that is, when $BA = AB$.

(c) Write $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$AX = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}, \quad XA = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}.$$

Comparing entries: $a + 2c = a$, $b + 2d = 2a + b$, $c = c$, $d = 2c + d$. So $c = 0$ and $d = a$, while a and b are arbitrary:

$$X = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = aI + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad a, b \in \mathbb{R}.$$

(For example B itself is not of this form, which matches $AB \neq BA$ from (a).)

Problem 9. ► Recitation

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

- (a) Find A^{-1} by Gauss–Jordan elimination on $[A \mid I]$, and check that $AA^{-1} = I$.
- (b) Use A^{-1} to solve $Ax = (2, 1, 5)$.
- (c) Run the same method on B . What happens, and what does it tell you?

Solution. (a) Row reduce $[A \mid I]$ until the left half is I ; the right half is then A^{-1} (Lecture 16).

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_1 \rightarrow R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right]. \end{aligned}$$

So $A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Check, row of A times column of A^{-1} : row 1: $(1, 2, 1)$ gives $3 + 0 - 2 = 1$, $-2 + 2 + 0 = 0$, $-1 + 0 + 1 = 0$; row 2: $(0, 1, 0)$ picks out the middle entries $0, 1, 0$; row 3: $(2, 4, 3)$ gives $6 + 0 - 6 = 0$, $-4 + 4 + 0 = 0$, $-2 + 0 + 3 = 1$. So $AA^{-1} = I$.

(b) $x = A^{-1}(2, 1, 5) = (6 - 2 - 5, 0 + 1 + 0, -4 + 0 + 5) = (-1, 1, 1)$. Check: $A(-1, 1, 1) = (-1 + 2 + 1, 0 + 1 + 0, -2 + 4 + 3) = (2, 1, 5)$.

(c) Start the same way on $[B \mid I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array} \right].$$

The left half has a zero row, so it can never become I : the third column has no pivot. Then $Bx = 0$ has a free variable and a nonzero solution, here $x = (1, -1, 1)$ (check: $1 - 2 + 1 = 0$, $2 - 5 + 3 = 0$, $1 - 3 + 2 = 0$). As in Problem 4(c), that rules out an inverse: B is not invertible.

Problem 10. Recall (Lecture 15): matrices cut into blocks are multiplied block by block, like matrices whose entries are matrices, as long as the block sizes match. Let A and D be invertible 2×2 matrices, B any 2×2 matrix, O the 2×2 zero matrix, and

$$M = \begin{bmatrix} A & B \\ O & D \end{bmatrix}.$$

- (a) Show that $\begin{bmatrix} A & B \\ O & D \end{bmatrix} \begin{bmatrix} P & X \\ O & Q \end{bmatrix} = \begin{bmatrix} AP & AX + BQ \\ O & DQ \end{bmatrix}$ for 2×2 blocks P, X, Q .

(b) Choose P, Q and X so that the product in (a) is I_4 , and check that the matrix you found is also an inverse of M from the left.

(c) Find M^{-1} as a 4×4 matrix for $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $B = I$ and $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and check one row of MM^{-1} .

Solution. (a) Block row times block column, keeping the order of the factors:

$$\begin{bmatrix} A & B \\ O & D \end{bmatrix} \begin{bmatrix} P & X \\ O & Q \end{bmatrix} = \begin{bmatrix} AP + BO & AX + BQ \\ OP + DO & OX + DQ \end{bmatrix} = \begin{bmatrix} AP & AX + BQ \\ O & DQ \end{bmatrix},$$

since a product with a zero block is zero.

(b) We need $AP = I$, $DQ = I$ and $AX + BQ = O$. So $P = A^{-1}$, $Q = D^{-1}$, and $AX = -BD^{-1}$, which gives $X = -A^{-1}BD^{-1}$ (multiply on the left by A^{-1}). The candidate is

$$N = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix}, \quad NM = \begin{bmatrix} A^{-1}A & A^{-1}B - A^{-1}BD^{-1}D \\ O & D^{-1}D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B - A^{-1}B \\ O & I \end{bmatrix} = I_4.$$

So M is invertible and $M^{-1} = N$.

(c) By the 2×2 formula, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ (here $ad - bc = 2 - 1 = 1$) and $D^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. With $B = I$,

$$X = -A^{-1}D^{-1} = - \begin{bmatrix} 1 \cdot 1 + (-1) \cdot 0 & 1 \cdot (-2) + (-1) \cdot 1 \\ -1 \cdot 1 + 2 \cdot 0 & (-1)(-2) + 2 \cdot 1 \end{bmatrix} = - \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -4 \end{bmatrix},$$

$$M = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & -1 & -1 & 3 \\ -1 & 2 & 1 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Row 1 of M , $(2, 1, 1, 0)$, times the four columns of M^{-1} : $2 - 1 + 0 + 0 = 1$, $-2 + 2 + 0 + 0 = 0$, $-2 + 1 + 1 + 0 = 0$, $6 - 4 - 2 + 0 = 0$. That is row 1 of I_4 .

C. Proof and concept

Problem 11. ► Recitation

True or false? Prove each true statement and give a counterexample for each false one. All matrices are real and $n \times n$ with $n \geq 2$.

- (a) If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear and $T(\mathbf{u}) = T(\mathbf{v})$ for some $\mathbf{u} \neq \mathbf{v}$, then $T(\mathbf{w}) = \mathbf{0}$ for some $\mathbf{w} \neq \mathbf{0}$.
- (b) If $AB = O$, then $A = O$ or $B = O$.
- (c) If $A^2 = I$, then $A = I$ or $A = -I$.
- (d) If AB is invertible, then A is invertible.

Solution. (a) True. Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$; then $\mathbf{w} \neq \mathbf{0}$ because $\mathbf{u} \neq \mathbf{v}$. By the two rules of linearity, $T(\mathbf{w}) = T(\mathbf{u} + (-1)\mathbf{v}) = T(\mathbf{u}) + (-1)T(\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$. $\square$

For example, the projection P of Problem 5 has $P(3, -1) = (2, 1) = P(2, 1)$, so it sends the difference $(1, -2)$ to $\mathbf{0}$.

(b) False. Take $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$. Neither is the zero matrix, but

$$AB = \begin{bmatrix} 2-2 & -4+4 \\ 4-4 & -8+8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(Each column of B is a multiple of $(2, -1)$, and $A(2, -1) = \mathbf{0}$.) Matrices can have “zero divisors”, unlike real numbers.

(c) False. The reflection $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ of Problem 7 satisfies

$$F^2 = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot (-1) \\ 0 \cdot 1 + (-1) \cdot 0 & 0 \cdot 0 + (-1)(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

but $F \neq I$ and $F \neq -I$. Geometrically: reflecting twice across the same line moves nothing. (For numbers, $a^2 = 1$ forces $a = \pm 1$, because $a^2 - 1 = (a - 1)(a + 1)$ and a product of real numbers is 0 only if a factor is 0. For matrices, $(A - I)(A + I) = O$ still holds, but by (b) that does not force a factor to be O .)

(d) True. Let $X = (AB)^{-1}$. By associativity, $I = (AB)X = A(BX)$. So A times the $n \times n$ matrix BX is I , and since one product is enough for square matrices (Lecture 16), A is invertible with $A^{-1} = BX$. $\square$

In the same way, $I = X(AB) = (XA)B$ shows that B is invertible, with $B^{-1} = XA$.

Problem 12. ► Recitation

A square matrix P is *idempotent* if $P^2 = P$. The projection of Problem 5 is one.

- (a) Prove that if P is idempotent, then so is $I - P$, and that $P(I - P) = (I - P)P = O$.
- (b) Prove that if P is idempotent, then $S = 2P - I$ satisfies $S^2 = I$.
- (c) Prove that the only invertible idempotent $n \times n$ matrix is I .
- (d) For the projection P of Problem 5, compute $Q = I - P$ and $Q\mathbf{x}$ for $\mathbf{x} = (3, -1)$, and describe Q geometrically.

Solution. Throughout we use the rules of matrix algebra (Lecture 14): the distributive laws, $IP = PI = P$, and $I^2 = I$.

(a) $(I - P)^2 = (I - P)(I - P) = I - P - P + P^2 = I - 2P + P = I - P$, using $P^2 = P$. So $I - P$ is idempotent. Also $P(I - P) = P - P^2 = P - P = O$ and $(I - P)P = P - P^2 = O$. $\square$

(b) $S^2 = (2P - I)(2P - I) = 4P^2 - 2P - 2P + I = 4P - 4P + I = I$. $\square$

(For the projection onto a line, S is the reflection across that line, and this is Problem 5(e) again, now for every idempotent at once.)

(c) Suppose P is invertible and $P^2 = P$. Multiply both sides on the left by P^{-1} and use associativity: $P^{-1}(PP) = P^{-1}P$, so $(P^{-1}P)P = I$, that is $IP = I$, and $P = I$. Conversely, I is invertible and $I^2 = I$. $\square$

So every idempotent other than I , such as a projection onto a line, is not invertible: it flattens the plane onto the line, and no map can undo that.

(d) $Q = I - \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, and $Q\mathbf{x} = \mathbf{x} - P\mathbf{x} = (3, -1) - (2, 1) = (1, -2)$. So $\mathbf{x} = P\mathbf{x} + Q\mathbf{x} = (2, 1) + (1, -2)$: the part along L plus the part perpendicular to L (indeed $(1, -2) \cdot (2, 1) = 0$). Q is the projection onto the line through the origin in the direction $(1, -2)$, perpendicular to L : by the formula of Problem 5 with $\mathbf{w} = (1, -2)$ and $\mathbf{w} \cdot \mathbf{w} = 5$, its columns are $\frac{1}{5}(1, -2)$ and $-\frac{2}{5}(1, -2)$, which are the

columns of Q . By (a), Q is idempotent and $PQ = O$: projecting onto L what was already projected onto the perpendicular line gives O .

Problem 13. Let A be an $n \times n$ matrix.

- (a) Suppose $A^2 = O$. Prove that $I - A$ is invertible, with inverse $I + A$. Check this for $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$.
- (b) Suppose $A^3 = O$. Find the inverse of $I - A$, and check your answer for $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.
- (c) Prove that if A is invertible and $AB = AC$, then $B = C$. Use Problem 11(b) to show that this can fail when A is not invertible.

Solution. (a) By the distributive laws, $(I - A)(I + A) = I + A - A - A^2 = I - O = I$, and $(I + A)(I - A) = I - A + A - A^2 = I$. So $I + A$ is an inverse of $I - A$. $\square$

For $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$: $A^2 = \begin{bmatrix} 4-4 & -8+8 \\ 2-2 & -4+4 \end{bmatrix} = O$, and $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3+4 & 4-4 \\ -3+3 & 4-3 \end{bmatrix} = I$.

(b) Try $I + A + A^2$, as for numbers, $\frac{1}{1-a} = 1 + a + a^2 + \dots$; for a matrix with $A^3 = O$ the series stops after three terms: $(I - A)(I + A + A^2) = I + A + A^2 - A - A^2 - A^3 = I - A^3 = I$, and in the same way $(I + A + A^2)(I - A) = I - A^3 = I$. So $(I - A)^{-1} = I + A + A^2$. $\square$

For the given A : $A^2 = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^3 = AA^2 = O$, so

$$(I - A)^{-1} = I + A + A^2 = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2-2 & 6-6 \\ 0 & 1 & 3-3 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

(c) Multiply $AB = AC$ on the left by A^{-1} : $A^{-1}(AB) = A^{-1}(AC)$, so by associativity $(A^{-1}A)B = (A^{-1}A)C$, that is $IB = IC$, and $B = C$. $\square$

Without invertibility: with A and B from Problem 11(b) and $C = O$, we have $AB = O = AC$ but $B \neq C$. (So this A has no inverse.)

D. Challenge and application

Problem 14. Computer graphics moves pictures by translations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{t}$, which are not linear when $\mathbf{t} \neq \mathbf{0}$ (Lecture 13). The trick is to write the point (x, y) of the plane as $(x, y, 1)$ in $\mathbb{R}^3$ (*homogeneous coordinates*), and to use the 3×3 matrices

$$M_{a,b} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \widehat{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) Show that $M_{a,b}$ sends $(x, y, 1)$ to $(x + a, y + b, 1)$, and that $\widehat{R}$ sends $(x, y, 1)$ to $(-y, x, 1)$, the rotation R of Problem 7.
- (b) Show that $M_{a,b}M_{c,d} = M_{a+c, b+d}$, and deduce that $M_{a,b}$ is invertible, with inverse $M_{-a, -b}$.
- (c) The rotation by 90° counterclockwise about the point $(1, 1)$ can be done in three steps: move $(1, 1)$ to the origin, rotate, move back. Write it as a product of three 3×3 matrices in the right order, and compute the product.

- (d) Find the images of the corners $(1, 1)$, $(2, 1)$, $(2, 2)$, $(1, 2)$ of a square, and draw the square and its image.

Solution. (a) $M_{a,b}(x, y, 1) = (x+0+a \cdot 1, 0+y+b \cdot 1, 0+0+1) = (x+a, y+b, 1)$, and $\widehat{R}(x, y, 1) = (-y, x, 1)$. So translations, which are not linear on $\mathbb{R}^2$, become matrices acting on the plane $z = 1$ of $\mathbb{R}^3$.

(b) Row times column:

$$M_{a,b}M_{c,d} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & c+a \\ 0 & 1 & d+b \\ 0 & 0 & 1 \end{bmatrix} = M_{a+c, b+d}.$$

Moving by (c, d) and then by (a, b) is moving by $(a+c, b+d)$. With $(c, d) = (-a, -b)$ this gives $M_{a,b}M_{-a,-b} = M_{0,0} = I$, and with the roles exchanged $M_{-a,-b}M_{a,b} = I$. So $M_{a,b}^{-1} = M_{-a,-b}$: the inverse of a move is the move back.

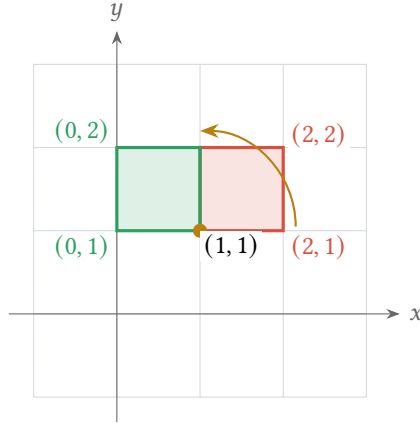
(c) The first step acts first, so it stands on the right (Lecture 14): the map is $M_{1,1}\widehat{R}M_{-1,-1}$.

$$\widehat{R}M_{-1,-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M = M_{1,1} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So the rotation about $(1, 1)$ is $(x, y) \mapsto (-y+2, x)$.

(d) $M(1, 1, 1) = (1, 1, 1)$: the centre stays put, as it should. $M(2, 1, 1) = (1, 2, 1)$, $M(2, 2, 1) = (0, 2, 1)$ and $M(1, 2, 1) = (0, 1, 1)$. So the corners go to $(1, 1)$, $(1, 2)$, $(0, 2)$, $(0, 1)$.



Red: the square. Green: its image after the quarter turn about the gold point $(1, 1)$. The corner $(2, 1)$ goes to $(1, 2)$, which is also a corner of the original square, and $(2, 2)$ goes to $(0, 2)$.