

Problem Sheet 4

Lectures 13 to 16: linear transformations of $\mathbb{R}^n$; matrix multiplication; matrix algebra; inverse matrices. Problems marked ► are solved in Recitation 4; the others are for practice on your own, with full solutions in the solutions booklet.

A. Warm-up

Problem 1. Recall (Lecture 13): a map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *linear* when $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{u})$ for all vectors $\mathbf{u}, \mathbf{v}$ and all scalars c . Decide whether each map is linear. For each linear map, give its standard matrix; for each map that is not linear, show one rule that fails, with numbers.

(a) $T(x, y) = (2x - y, 3y)$

(b) $T(x, y) = (x + y, 2)$

(c) $T(x, y) = (x, |y|)$

(d) $T(x, y, z) = (x - z, 0)$

Problem 2. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$.

(a) Compute AB and BA . Do A and B commute?

(b) Compute AC . What is its size?

(c) Explain why CA is not defined.

Problem 3. Let $M = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 5 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 3 & -1 \\ 6 & 1 & 2 \end{bmatrix}$.

(a) Compute MN and $(MN)^T$. Then compute $N^T M^T$ and $M^T N^T$. Which of the two equals $(MN)^T$?

(b) Write A as the sum of a symmetric matrix S and a skew-symmetric matrix K (one with $K^T = -K$).

Problem 4. Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$.

(a) Find A^{-1} with the formula for 2×2 inverses (Lecture 16), and check that $AA^{-1} = I$.

(b) Use A^{-1} to solve $A\mathbf{x} = (5, 1)$.

(c) Show that B is not invertible.

B. Core

Problem 5. ► Recitation

Let L be the line through the origin in the direction $\mathbf{u} = (2, 1)$. The *projection* onto L sends each $\mathbf{x}$ in $\mathbb{R}^2$ to the point of L closest to it,

$$P(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u},$$

and the *reflection* across L is $S(\mathbf{x}) = 2P(\mathbf{x}) - \mathbf{x}$.

- Find the standard matrix of P by computing $P(\mathbf{e}_1)$ and $P(\mathbf{e}_2)$.
- Find the standard matrix of S .
- Compute $P(\mathbf{x})$ and $S(\mathbf{x})$ for $\mathbf{x} = (3, -1)$, and check that $S(\mathbf{x})$ has the same length as $\mathbf{x}$.
- Draw L , $\mathbf{x}$, $P(\mathbf{x})$ and $S(\mathbf{x})$.
- Explain geometrically why $P^2 = P$ and $S^2 = I$, and check $S^2 = I$ by multiplying the matrices.

Problem 6. ► Recitation

A linear map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ satisfies $T(1, 1) = (1, 2, 0)$ and $T(1, -1) = (3, 0, 2)$.

- Write $\mathbf{e}_1$ and $\mathbf{e}_2$ as combinations of $(1, 1)$ and $(1, -1)$, and find $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.
- Find the standard matrix of T and compute $T(5, 1)$.
- Check $T(5, 1)$ a second way, by writing $(5, 1)$ as a combination of $(1, 1)$ and $(1, -1)$.
- Is there a linear map with $T(1, 1) = (1, 2, 0)$ and $T(2, 2) = (2, 5, 0)$? Explain.

Problem 7. ► Recitation

Let R be the rotation of the plane by 90° counterclockwise and F the reflection across the x_1 -axis.

- Write down the standard matrices of R and F (Lecture 13), from the images of $\mathbf{e}_1$ and $\mathbf{e}_2$.
- Compute RF and FR . Which map does each product stand for (first which map, then which)? Do R and F commute?
- Describe RF and FR geometrically, and draw $\mathbf{v} = (3, 1)$, $RF\mathbf{v}$ and $FR\mathbf{v}$.
- Compute R^2 and R^4 , and explain the answers geometrically.

Problem 8. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

- Compute $(A + B)^2$ and $A^2 + 2AB + B^2$. Are they equal?
- Expand $(A + B)^2 = (A + B)(A + B)$ with the distributive laws, and check your expansion with these A and B . When is $(A + B)^2 = A^2 + 2AB + B^2$?
- Find all 2×2 matrices X with $AX = XA$.

Problem 9. ► Recitation

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

- (a) Find A^{-1} by Gauss–Jordan elimination on $[A \mid I]$, and check that $AA^{-1} = I$.
- (b) Use A^{-1} to solve $A\mathbf{x} = (2, 1, 5)$.
- (c) Run the same method on B . What happens, and what does it tell you?

Problem 10. Recall (Lecture 15): matrices cut into blocks are multiplied block by block, like matrices whose entries are matrices, as long as the block sizes match. Let A and D be invertible 2×2 matrices, B any 2×2 matrix, O the 2×2 zero matrix, and

$$M = \begin{bmatrix} A & B \\ O & D \end{bmatrix}.$$

- (a) Show that $\begin{bmatrix} A & B \\ O & D \end{bmatrix} \begin{bmatrix} P & X \\ O & Q \end{bmatrix} = \begin{bmatrix} AP & AX + BQ \\ O & DQ \end{bmatrix}$ for 2×2 blocks P, X, Q .
- (b) Choose P, Q and X so that the product in (a) is I_4 , and check that the matrix you found is also an inverse of M from the left.
- (c) Find M^{-1} as a 4×4 matrix for $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $B = I$ and $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and check one row of MM^{-1} .

C. Proof and concept

Problem 11. ► Recitation

True or false? Prove each true statement and give a counterexample for each false one. All matrices are real and $n \times n$ with $n \geq 2$.

- (a) If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear and $T(\mathbf{u}) = T(\mathbf{v})$ for some $\mathbf{u} \neq \mathbf{v}$, then $T(\mathbf{w}) = \mathbf{0}$ for some $\mathbf{w} \neq \mathbf{0}$.
- (b) If $AB = O$, then $A = O$ or $B = O$.
- (c) If $A^2 = I$, then $A = I$ or $A = -I$.
- (d) If AB is invertible, then A is invertible.

Problem 12. ► Recitation

A square matrix P is *idempotent* if $P^2 = P$. The projection of Problem 5 is one.

- (a) Prove that if P is idempotent, then so is $I - P$, and that $P(I - P) = (I - P)P = O$.
- (b) Prove that if P is idempotent, then $S = 2P - I$ satisfies $S^2 = I$.
- (c) Prove that the only invertible idempotent $n \times n$ matrix is I .
- (d) For the projection P of Problem 5, compute $Q = I - P$ and $Q\mathbf{x}$ for $\mathbf{x} = (3, -1)$, and describe Q geometrically.

Problem 13. Let A be an $n \times n$ matrix.

- (a) Suppose $A^2 = O$. Prove that $I - A$ is invertible, with inverse $I + A$. Check this for $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$.
- (b) Suppose $A^3 = O$. Find the inverse of $I - A$, and check your answer for $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.
- (c) Prove that if A is invertible and $AB = AC$, then $B = C$. Use Problem 11(b) to show that this can fail when A is not invertible.

D. Challenge and application

Problem 14. Computer graphics moves pictures by translations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{t}$, which are not linear when $\mathbf{t} \neq \mathbf{0}$ (Lecture 13). The trick is to write the point (x, y) of the plane as $(x, y, 1)$ in $\mathbb{R}^3$ (*homogeneous coordinates*), and to use the 3×3 matrices

$$M_{a,b} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \widehat{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) Show that $M_{a,b}$ sends $(x, y, 1)$ to $(x + a, y + b, 1)$, and that $\widehat{R}$ sends $(x, y, 1)$ to $(-y, x, 1)$, the rotation R of Problem 7.
- (b) Show that $M_{a,b}M_{c,d} = M_{a+c, b+d}$, and deduce that $M_{a,b}$ is invertible, with inverse $M_{-a, -b}$.
- (c) The rotation by 90° counterclockwise about the point $(1, 1)$ can be done in three steps: move $(1, 1)$ to the origin, rotate, move back. Write it as a product of three 3×3 matrices in the right order, and compute the product.
- (d) Find the images of the corners $(1, 1)$, $(2, 1)$, $(2, 2)$, $(1, 2)$ of a square, and draw the square and its image.