

Problem Sheet 5: Solutions

Lectures 17 to 20: the invertible matrix theorem; LU factorisation; subspaces of $\mathbb{R}^n$; basis and dimension in $\mathbb{R}^n$. Problems marked ► are solved in Recitation 5; the others are for practice on your own, with full solutions in the solutions booklet.

A. Warm-up

Problem 1. Recall (Lecture 17): an *elementary matrix* E is what I becomes after one row operation, and EA performs that same row operation on A . Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}.$$

- Which row operation does each of E_1 , E_2 , E_3 perform?
- Compute E_1A , E_2A and E_3A , and check that each one is A after that row operation.
- Write down E_1^{-1} , E_2^{-1} and E_3^{-1} without any calculation, by asking which row operation undoes each one.

Solution. (a) E_1 is I after $R_3 \rightarrow R_3 - 3R_1$; E_2 is I after $R_1 \leftrightarrow R_2$; E_3 is I after $R_3 \rightarrow -2R_3$.

(b) Row by row:

$$E_1A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 3-3 & 4-6 & 2+3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 5 \end{bmatrix}, \quad E_2A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}, \quad E_3A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ -6 & -8 & -4 \end{bmatrix}.$$

For example, row 3 of E_1A is row 3 of E_1 , $(-3, 0, 1)$, times the columns of A : $-3 \cdot 1 + 0 + 3 = 0$, $-3 \cdot 2 + 0 + 4 = -2$, $-3 \cdot (-1) + 0 + 2 = 5$. That is $R_3 - 3R_1$, as claimed. In the same way E_2A is A with rows 1 and 2 swapped, and E_3A is A with row 3 multiplied by -2 .

(c) $R_3 \rightarrow R_3 - 3R_1$ is undone by $R_3 \rightarrow R_3 + 3R_1$; a swap is undone by the same swap; and $R_3 \rightarrow -2R_3$ is undone by $R_3 \rightarrow -\frac{1}{2}R_3$. So

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = E_2, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Check one: $E_1^{-1}E_1$ has row 3 equal to $(3, 0, 1)E_1 = (3 - 3, 0, 1) = (0, 0, 1)$, and rows 1 and 2 of I ; so $E_1^{-1}E_1 = I$.

Problem 2. Recall (Lecture 19): a *subspace* of $\mathbb{R}^n$ is a subset H that contains $\mathbf{0}$ and is closed under addition and under multiplication by scalars. Which of these sets are subspaces? Prove your answer for each subspace, and show one rule that fails, with numbers, for each set that is not.

- $H_1 = \{(x, y, z) \in \mathbb{R}^3 : x + 3y - 2z = 0\}$
- $H_2 = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$
- $H_3 = \{(x, y) \in \mathbb{R}^2 : y = |x|\}$
- $H_4 = \{(t, 2t, -t) : t \in \mathbb{R}\}$

Solution. (a) A subspace. $\mathbf{0} \in H_1$ since $0 + 0 - 0 = 0$. If (x, y, z) and (x', y', z') are in H_1 , their sum satisfies $(x + x') + 3(y + y') - 2(z + z') = (x + 3y - 2z) + (x' + 3y' - 2z') = 0 + 0 = 0$, and for a scalar c , $cx + 3cy - 2cz = c(x + 3y - 2z) = c \cdot 0 = 0$. (Also: $H_1 = \text{Nul} \begin{bmatrix} 1 & 3 & -2 \end{bmatrix}$, and every null space is a subspace.)

(b) Not a subspace: $\mathbf{0} \notin H_2$, because $0 + 0 + 0 = 0 \neq 1$.

(c) Not a subspace. It contains $\mathbf{0}$ (as $0 = |0|$), but it is not closed under scalar multiples: $(1, 1) \in H_3$, while $(-1)(1, 1) = (-1, -1)$ is not, since $|-1| = 1 \neq -1$. (Addition fails too: $(1, 1) + (-1, 1) = (0, 2)$, and $2 \neq |0|$.) H_3 is a V shape, made of two half-lines.

(d) A subspace: $H_4 = \{t(1, 2, -1)\} = \text{Span}\{(1, 2, -1)\}$, a line through the origin, and every span is a subspace. Directly: $\mathbf{0}$ is the case $t = 0$, $t(1, 2, -1) + s(1, 2, -1) = (t + s)(1, 2, -1)$, and $c(t(1, 2, -1)) = (ct)(1, 2, -1)$.

Problem 3. Recall (Lecture 19): $\text{Nul } A$ is the set of solutions of $A\mathbf{x} = \mathbf{0}$, and $\text{Col } A$ is the span of the columns of A . Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 0 & 3 & -3 \end{bmatrix}.$$

(a) Is $\mathbf{u} = (2, -2, -2)$ in $\text{Nul } A$? Is $\mathbf{w} = (1, 1, 0)$?

(b) Is $\mathbf{b} = (1, 5, 3)$ in $\text{Col } A$? Is $\mathbf{c} = (1, 0, 0)$? When the answer is yes, write the vector as a combination of the columns of A .

Solution. (a) Membership in $\text{Nul } A$ is a direct check: multiply. $A\mathbf{u} = (2+2-4, 4-2-2, 0-6+6) = (0, 0, 0)$, so $\mathbf{u} \in \text{Nul } A$. $A\mathbf{w} = (1-1+0, 2+1+0, 0+3+0) = (0, 3, 3) \neq \mathbf{0}$, so $\mathbf{w} \notin \text{Nul } A$.

(b) Membership in $\text{Col } A$ is a question about a system: $\mathbf{b} \in \text{Col } A$ exactly when $A\mathbf{x} = \mathbf{b}$ has a solution. Row reduce $[A \mid \mathbf{b}]$ and $[A \mid \mathbf{c}]$ together, with $R_2 \rightarrow R_2 - 2R_1$, then $R_3 \rightarrow R_3 - R_2$:

$$\left[\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 5 & 0 \\ 0 & 3 & -3 & 3 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & 1 \\ 0 & 3 & -3 & 3 & -2 \\ 0 & 3 & -3 & 3 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & 1 \\ 0 & 3 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 & 2 \end{array} \right].$$

For $\mathbf{c}$ the last row reads $0 = 2$: no solution, so $\mathbf{c} \notin \text{Col } A$. For $\mathbf{b}$ the system is consistent, with x_3 free. Taking $x_3 = 0$: $3x_2 = 3$, so $x_2 = 1$, and $x_1 - 1 = 1$, so $x_1 = 2$. Hence

$$\mathbf{b} = 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 4+1 \\ 0+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}.$$

(Other combinations work too, because $\mathbf{u} \in \text{Nul } A$: adding a multiple of $\mathbf{u}$ to the weights does not change the result.)

Problem 4. Recall (Lecture 18): if $A = LU$, then $A\mathbf{x} = \mathbf{b}$ is solved in two triangular steps: first $L\mathbf{y} = \mathbf{b}$ by forward substitution, then $U\mathbf{x} = \mathbf{y}$ by back substitution. Let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}.$$

(a) Solve $A\mathbf{x} = \mathbf{b}$ for $A = LU$, without computing A .

(b) Compute $A = LU$, and check your answer to (a).

Solution. (a) Forward substitution in $Ly = \mathbf{b}$: $y_1 = -1$; $2y_1 + y_2 = -2$ gives $y_2 = -2 + 2 = 0$; $-y_1 + 3y_2 + y_3 = 4$ gives $y_3 = 4 - 1 - 0 = 3$. So $\mathbf{y} = (-1, 0, 3)$.

Back substitution in $U\mathbf{x} = \mathbf{y}$: $3x_3 = 3$, so $x_3 = 1$; $x_2 + 2x_3 = 0$, so $x_2 = -2$; $2x_1 + x_2 - x_3 = -1$, so $2x_1 = -1 + 2 + 1 = 2$ and $x_1 = 1$. So $\mathbf{x} = (1, -2, 1)$.

(b) Row i of LU is row i of L times U , a combination of the rows of U : row 1 is $(2, 1, -1)$; row 2 is $2(2, 1, -1) + (0, 1, 2) = (4, 3, 0)$; row 3 is $-(2, 1, -1) + 3(0, 1, 2) + (0, 0, 3) = (-2, 2, 10)$. So

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 3 & 0 \\ -2 & 2 & 10 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - 2 - 1 \\ 4 - 6 + 0 \\ -2 - 4 + 10 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix} = \mathbf{b}.$$

B. Core

Problem 5. ► Recitation

Use the invertible matrix theorem (Lecture 17, Theorem 3.6.10) to decide which of these matrices are invertible. Do not compute any inverse; name the statement of the theorem that you use.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & -1 \\ 0 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -2 & 0 \\ 4 & 5 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 1 & 1 & -2 \end{bmatrix}.$$

(a) A , (b) B , (c) C ;

(d) an $n \times n$ matrix D about which we only know that $D^2 = D + I$;

(e) all numbers h for which $M_h = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & h \\ 2 & 3 & 5 \end{bmatrix}$ is invertible.

Solution. (a) A : row 3 is $2 \cdot (\text{row } 2) - (\text{row } 1)$, since $2(1, 3, -1) - (2, 1, 4) = (0, 5, -6)$. So the two replacements $R_3 \rightarrow R_3 - 2R_2$ and $R_3 \rightarrow R_3 + R_1$ turn row 3 into a zero row. An echelon form of A then has at most 2 pivots, fewer than 3, so statement (4), “ A has 3 pivot positions”, fails, and A is *not invertible*.

(b) B is lower triangular, so $B^T = \begin{bmatrix} 3 & 1 & 4 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ is upper triangular. That is already an echelon form, with pivots 3, -2 , 1 in all three columns. So B^T has 3 pivot positions and is invertible. That is statement (11) for B , so B is *invertible*.

(c) Each row of C adds up to 0: $2 - 1 - 1 = 0$, $0 + 3 - 3 = 0$, $1 + 1 - 2 = 0$. That says exactly $C(1, 1, 1) = \mathbf{0}$. So $C\mathbf{x} = \mathbf{0}$ has a nontrivial solution, statement (3) fails, and C is *not invertible*.

(d) $D^2 = D + I$ gives $D^2 - D = I$, that is $D(D - I) = I$ by the distributive law. So statement (10) holds for D (the matrix called D in the theorem is here $D - I$), and D is *invertible*, with $D^{-1} = D - I$. The argument needs nothing about the entries of D . (For example $D = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $D^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = D + I$, and indeed $D(D - I) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.)

(e) Row reduce: $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - 2R_1$, then $R_3 \rightarrow R_3 - R_2$:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & h \\ 2 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & h-2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & h-2 \\ 0 & 0 & 3-h \end{bmatrix}.$$

There are three pivot positions (statement (4)) exactly when $3 - h \neq 0$. So M_h is invertible for every $h \neq 3$, and not invertible for $h = 3$. (For $h = 3$, row 3 of M_3 is the sum of rows 1 and 2.)

Problem 6. ► Recitation

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 7 \\ -2 & 5 & 1 \end{bmatrix}$.

- (a) Find an LU factorisation of A (Lecture 18): reduce A to an echelon form U using only row replacements, and record the multipliers in L . Check that $LU = A$.
- (b) Use L and U to solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (4, 6, -6)$ and for $\mathbf{b} = (2, 8, 8)$.
- (c) Explain why $B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ has no LU factorisation, and find a permutation matrix P and matrices L, U with $PB = LU$.

Solution. (a) The first pivot is 2. $R_2 \rightarrow R_2 - 2R_1$ (multiplier $l_{21} = 2$) and $R_3 \rightarrow R_3 + R_1$ (multiplier $l_{31} = -1$):

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 7 \\ -2 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 1 \\ 0 & 6 & 4 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} = U,$$

with $l_{32} = 2$. So

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}.$$

Check, row by row: row 2 of LU is $2(2, 1, 3) + (0, 3, 1) = (4, 5, 7)$, and row 3 is $-(2, 1, 3) + 2(0, 3, 1) + (0, 0, 2) = (-2, 5, 1)$. Row 1 is $(2, 1, 3)$. So $LU = A$.

(b) For $\mathbf{b} = (4, 6, -6)$. Forward: $y_1 = 4$, $y_2 = 6 - 2 \cdot 4 = -2$, $y_3 = -6 + 4 - 2(-2) = 2$. Back: $2x_3 = 2$, so $x_3 = 1$; $3x_2 + 1 = -2$, so $x_2 = -1$; $2x_1 - 1 + 3 = 4$, so $x_1 = 1$. Thus $\mathbf{x} = (1, -1, 1)$. Check: $A\mathbf{x} = (2 - 1 + 3, 4 - 5 + 7, -2 - 5 + 1) = (4, 6, -6)$.

For $\mathbf{b} = (2, 8, 8)$. Forward: $y_1 = 2$, $y_2 = 8 - 4 = 4$, $y_3 = 8 + 2 - 2 \cdot 4 = 2$. Back: $x_3 = 1$; $3x_2 + 1 = 4$, so $x_2 = 1$; $2x_1 + 1 + 3 = 2$, so $x_1 = -1$. Thus $\mathbf{x} = (-1, 1, 1)$. Check: $A\mathbf{x} = (-2 + 1 + 3, -4 + 5 + 7, 2 + 5 + 1) = (2, 8, 8)$.

The factorisation is done once; each new right-hand side costs only the two triangular solves.

(c) Elimination cannot start: the entry in the pivot position $(1, 1)$ is 0, and the replacements of LU elimination only change rows below the pivot row, so row 1 never changes. In terms of matrices: if $B = LU$ with L unit lower triangular, then row 1 of B is row 1 of U , so $u_{11} = 0$. The $(2, 1)$ entry of LU is $l_{21}u_{11} + 1 \cdot u_{21} = 0 + 0 = 0$ (as $u_{11} = 0$, the leading entry of row 1 of the echelon form U lies to the right of column 1, so column 1 of U is zero), but $b_{21} = 1$. A contradiction.

Swap rows 1 and 2 first. With $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$PB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 4 & 3 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = U, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

Check: row 3 of LU is $2(1, 1, 0) + (0, 2, 1) + (0, 0, 2) = (2, 4, 3)$, and rows 1 and 2 are those of U . So $PB = LU$.

Problem 7. ► Recitation

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ -1 & -2 & 1 & -5 \end{bmatrix}.$$

- Find the reduced row echelon form R of A .
- Find a basis of $\text{Nul } A$ (Lecture 20), and its dimension.
- Find a basis of $\text{Col } A$, and its dimension. Is $\text{Col } A$ equal to $\text{Col } R$?
- Is $\mathbf{b} = (1, 3, 0)$ in $\text{Col } A$? Is $\mathbf{e}_1 = (1, 0, 0)$?

Solution. (a) $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 + R_1$, then $R_3 \rightarrow R_3 - R_2$:

$$A \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

The pivots are in columns 1 and 3, and this is already reduced.

(b) $A\mathbf{x} = \mathbf{0}$ has the same solutions as $R\mathbf{x} = \mathbf{0}$: $x_1 + 2x_2 + 3x_4 = 0$ and $x_3 - 2x_4 = 0$, with $x_2 = s$ and $x_4 = t$ free. So

$$\mathbf{x} = \begin{bmatrix} -2s - 3t \\ s \\ 2t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

The two vectors span $\text{Nul } A$, and they are independent: in entries 2 and 4 (the free variables), one has 1, 0 and the other 0, 1, so $s(\cdot) + t(\cdot) = \mathbf{0}$ forces $s = t = 0$. So $((-2, 1, 0, 0), (-3, 0, 2, 1))$ is a basis of $\text{Nul } A$, and $\dim \text{Nul } A = 2$, the number of free variables. Check: $A(-2, 1, 0, 0) = (-2 + 2, -4 + 4, 2 - 2) = \mathbf{0}$ and $A(-3, 0, 2, 1) = (-3 + 0 + 0 + 3, -6 + 2 + 4, 3 + 2 - 5) = \mathbf{0}$.

(c) The pivot columns of A form a basis of $\text{Col } A$: $\mathbf{a}_1 = (1, 2, -1)$ and $\mathbf{a}_3 = (0, 1, 1)$. So $\dim \text{Col } A = 2$, the number of pivots. The other columns are combinations of these, with the weights read off from R : $\mathbf{a}_2 = 2\mathbf{a}_1$ and $\mathbf{a}_4 = 3\mathbf{a}_1 - 2\mathbf{a}_3 = (3 - 0, 6 - 2, -3 - 2) = (3, 4, -5)$.

$\text{Col } R$ is not $\text{Col } A$: every column of R has third entry 0, so $\text{Col } R$ is the plane $x_3 = 0$, while $\mathbf{a}_1 \in \text{Col } A$ has third entry -1 . Row operations keep the relations between the columns, but change the columns themselves; so the basis is taken from A , in the positions of the pivots of R .

(d) $\text{Col } A$ is the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_3$. Try $\mathbf{b} = x\mathbf{a}_1 + y\mathbf{a}_3 = (x, 2x + y, -x + y)$: the first entry gives $x = 1$, the second $y = 3 - 2 = 1$, and then the third is $-1 + 1 = 0$, as required. So $\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_3 \in \text{Col } A$. For $\mathbf{e}_1$: the first entry gives $x = 1$, the second $y = -2$, and then the third would be $-1 - 2 = -3 \neq 0$. So $\mathbf{e}_1 \notin \text{Col } A$, although $\mathbf{e}_1$ is the first column of R . (Another way: $\mathbf{n} = (3, -1, 1)$ is perpendicular to $\mathbf{a}_1$ and $\mathbf{a}_3$, so $\text{Col } A$ is the plane $3x_1 - x_2 + x_3 = 0$; $\mathbf{b}$ satisfies it ($3 - 3 + 0 = 0$), $\mathbf{e}_1$ does not ($3 \neq 0$)).

Problem 8. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$.

- Reduce A to I with three row operations: a swap, a scaling and a replacement. Write down the elementary matrices E_1, E_2, E_3 of these operations, in order, so that $E_3E_2E_1A = I$.
- Deduce A^{-1} as a product of elementary matrices, compute it, and check that $AA^{-1} = I$.
- Write A itself as a product of elementary matrices (Lecture 17), and check the product.

Solution. (a) $R_1 \leftrightarrow R_2$ gives $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$; $R_2 \rightarrow \frac{1}{2}R_2$ gives $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; $R_1 \rightarrow R_1 - 3R_2$ gives I . The elementary matrices are I after each operation:

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

(b) $E_3E_2E_1A = I$ says that $E_3E_2E_1$ is a left inverse of the square matrix A , so $A^{-1} = E_3E_2E_1$ (one product is enough, Lecture 16):

$$E_2E_1 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad A^{-1} = E_3(E_2E_1) = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Check: $AA^{-1} = \begin{bmatrix} 0+1 & 0+0 \\ -\frac{3}{2}+\frac{3}{2} & 1+0 \end{bmatrix} = I$. (The 2×2 formula agrees: $ad - bc = 0 - 2 = -2$ and $-\frac{1}{2} \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3/2 & 1 \\ 1/2 & 0 \end{bmatrix}$.)

(c) Invert both sides of $A^{-1} = E_3E_2E_1$, reversing the order: $A = E_1^{-1}E_2^{-1}E_3^{-1}$, where each inverse undoes its operation:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Check: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, and swapping its rows gives $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = A$.

Problem 9. ► Recitation

Let $H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + 2x_2 - x_3 = 0\}$, a plane through the origin.

- Write H as the null space of a matrix, find a basis of H , and give $\dim H$.
- Check that $\mathbf{v} = (1, 1, 3)$ lies in H , and write it in terms of your basis.
- Exchange one vector of your basis for $\mathbf{v}$, so that the new list is still a basis of H . Justify that it is one.
- Extend your basis of (a) to a basis of $\mathbb{R}^3$.
- Is $((-2, 1, 0), (1, 0, 1), (1, 1, 3))$ a basis of $\mathbb{R}^3$?

Solution. (a) $H = \text{Nul} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$. The matrix is already reduced, with its pivot in column 1 and $x_2 = s$, $x_3 = t$ free; so $x_1 = -2s + t$ and

$$\mathbf{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$\mathbf{b}_1 = (-2, 1, 0)$ and $\mathbf{b}_2 = (1, 0, 1)$ span H , and they are independent (look at entries 2 and 3, those of the free variables). So $(\mathbf{b}_1, \mathbf{b}_2)$ is a basis of H , and $\dim H = 2$.

(b) $1+2 \cdot 1-3 = 0$, so $\mathbf{v} \in H$. Its free entries are $x_2 = 1$ and $x_3 = 3$, so $\mathbf{v} = 1\mathbf{b}_1 + 3\mathbf{b}_2 = (-2+3, 1+0, 0+3) = (1, 1, 3)$.

(c) In $\mathbf{v} = 1\mathbf{b}_1 + 3\mathbf{b}_2$ both weights are nonzero, so $\mathbf{v}$ can replace either vector. Replace $\mathbf{b}_1$: the list $(\mathbf{v}, \mathbf{b}_2) = ((1, 1, 3), (1, 0, 1))$. It spans H , because $\mathbf{b}_1 = \mathbf{v} - 3\mathbf{b}_2$ is in its span, so every combination of $\mathbf{b}_1, \mathbf{b}_2$ is a combination of $\mathbf{v}, \mathbf{b}_2$. It is independent, because neither vector is a multiple of the other (the second entries are 1 and 0, while $\mathbf{b}_2 \neq \mathbf{0}$). So it is a basis of H . (Alternatively: two independent vectors in a subspace of dimension 2 always form a basis.)

(d) Add a vector outside H , for example $\mathbf{e}_1 = (1, 0, 0)$ ($1+0-0 = 1 \neq 0$). The list $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{e}_1)$ is independent, because a vector outside the span of an independent list keeps it independent. Three

independent vectors in $\mathbb{R}^3$, which has dimension 3, form a basis. Directly, with the three vectors as columns:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2, R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(the last step is $R_3 \rightarrow R_3 + 2R_1$, then $R_3 \rightarrow R_3 - R_2$), so there is a pivot in every column.

(e) No. All three vectors lie in the plane H (part (b)), and $(1, 1, 3) = (-2, 1, 0) + 3(1, 0, 1)$, so the list is dependent. Three vectors from a 2-dimensional subspace can never be independent.

Problem 10. ► Recitation

Let $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$.

- Find a basis and the dimension of $\text{Nul } A$ and of $\text{Col } A$.
- Is A invertible? Name two statements of the invertible matrix theorem that fail.
- Find all solutions of $A\mathbf{x} = (3, -6)$, and explain why $A\mathbf{x} = (1, 1)$ has none.
- Draw two planes side by side: in the input plane, $\text{Nul } A$ and the solution set of $A\mathbf{x} = (3, -6)$; in the output plane, $\text{Col } A$, the vector $(3, -6)$ and the vector $(1, 1)$.

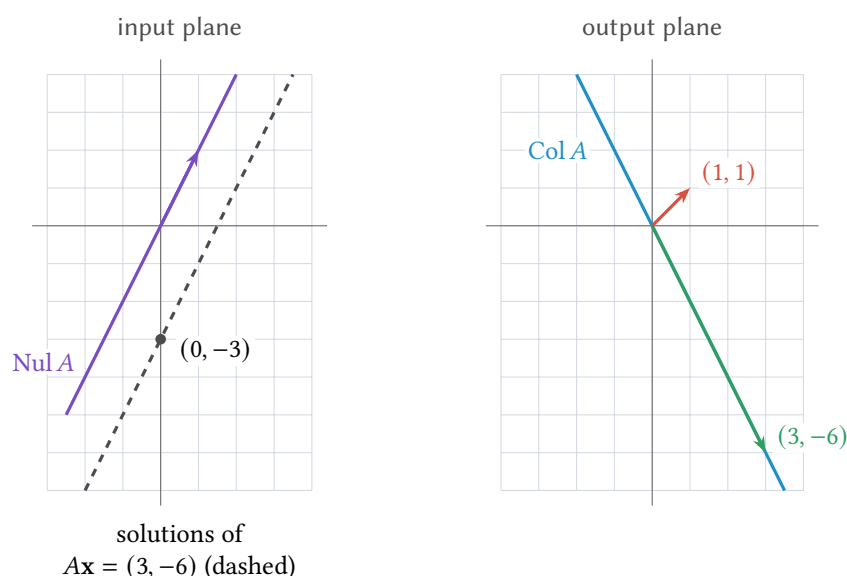
Solution. (a) $R_2 \rightarrow R_2 + 2R_1$ gives $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$: one pivot, in column 1, and $x_2 = t$ free. Then $2x_1 = t$, so $\mathbf{x} = t(\frac{1}{2}, 1)$, and with $t = 2s$, $\mathbf{x} = s(1, 2)$. So $\text{Nul } A = \text{Span}\{(1, 2)\}$, with basis $((1, 2))$ and dimension 1. The pivot column $(2, -4)$ is a basis of $\text{Col } A$; it is a multiple of $(1, -2)$, so $\text{Col } A = \text{Span}\{(1, -2)\}$, of dimension 1. (The second column $(-1, 2)$ is $-\frac{1}{2}$ times the first.)

(b) No. For example, statement (3) fails, since $A\mathbf{x} = \mathbf{0}$ has the nontrivial solution $(1, 2)$; and statement (8) fails, since the columns span only a line, not $\mathbb{R}^2$. By the theorem all the other statements fail too: for instance, A has only one pivot position, and its columns are dependent.

(c) $A\mathbf{x} = (3, -6)$: the second equation is -2 times the first, so the system is the single equation $2x_1 - x_2 = 3$. Taking $x_1 = 0$ gives one solution, $\mathbf{p} = (0, -3)$; adding any vector of $\text{Nul } A$ gives all of them: $\mathbf{x} = (0, -3) + s(1, 2)$, $s \in \mathbb{R}$: a line parallel to $\text{Nul } A$. Check: $A(0, -3) = (0 + 3, 0 - 6) = (3, -6)$ and $A(1, 2) = \mathbf{0}$.

$A\mathbf{x} = (1, 1)$ has no solution because $(1, 1) \notin \text{Col } A$: every vector of $\text{Col } A$ is $c(1, -2)$, whose second entry is -2 times its first, and $1 \neq -2$. (Row reduction gives the row $[0 \ 0 \ 3]$.)

(d) On the left, the input plane: $\text{Nul } A$ (violet, with the basis vector $(1, 2)$ drawn as an arrow) and the solution set of $Ax = (3, -6)$ (dashed), a parallel line through $(0, -3)$. On the right, the output plane: $\text{Col } A$ (blue), with $(3, -6)$ on it (green) and $(1, 1)$ off it (coral).



Every vector of the input plane is sent into the blue line $\text{Col } A$. The whole dashed line is sent to the single green vector $(3, -6)$, and $\text{Nul } A$ to $\mathbf{0}$. The coral vector $(1, 1)$ is not on the blue line, so nothing is sent to it.

C. Proof and concept

Problem 11. ► Recitation

True or false? Prove each true statement and give a counterexample for each false one.

- If A is a 3×3 matrix and the systems $Ax = \mathbf{e}_1$ and $Ax = \mathbf{e}_2$ both have solutions, then A is invertible.
- If H is a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ is a vector not in H , then the set $\mathbf{v} + H = \{\mathbf{v} + \mathbf{h} : \mathbf{h} \in H\}$ is not a subspace.
- Every square matrix that has an LU factorisation is invertible.
- If $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a basis of $\mathbb{R}^3$, then so is $(\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w})$.

Solution. (a) False. Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $A\mathbf{e}_1 = \mathbf{e}_1$ and $A\mathbf{e}_2 = \mathbf{e}_2$, so both systems have solutions; but the third column of A is zero, so $A\mathbf{e}_3 = \mathbf{0}$ and $Ax = \mathbf{0}$ has a nontrivial solution. A is not invertible.

If $Ax = \mathbf{e}_3$ has a solution as well, then A is invertible. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be the three solutions and X the matrix with these columns. Column j of AX is $A\mathbf{x}_j = \mathbf{e}_j$, so $AX = I$, and A is invertible by statement (10) of the invertible matrix theorem.

(b) True. Suppose $\mathbf{v} + H$ were a subspace. Then it would contain $\mathbf{0}$, so $\mathbf{0} = \mathbf{v} + \mathbf{h}$ for some $\mathbf{h} \in H$. Then $\mathbf{v} = -\mathbf{h} = (-1)\mathbf{h}$, which is in H because H is closed under scalar multiples. That contradicts $\mathbf{v} \notin H$. $\square$

(For example, the solution set of $Ax = \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$ is $\mathbf{p} + \text{Nul } A$, and it is not a subspace: $\mathbf{p} \notin \text{Nul } A$ because $A\mathbf{p} = \mathbf{b} \neq \mathbf{0}$.)

(c) False. For example,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

an LU factorisation (check: row 2 of the product is $2(1, 2) + (0, 0) = (2, 4)$). The matrix is not invertible: its second column is twice the first. An LU factorisation only records elimination without swaps; whether the matrix is invertible depends on whether U has a pivot in every column.

(d) True. Three vectors in $\mathbb{R}^3$ form a basis as soon as they are independent ($\dim \mathbb{R}^3 = 3$, Lecture 20). Suppose $au + b(u + v) + c(u + v + w) = \mathbf{0}$. Collecting terms, $(a + b + c)u + (b + c)v + cw = \mathbf{0}$. Since u, v, w are independent, all three weights are 0: $c = 0$, then $b = 0$, then $a = 0$. So the new list is independent, and therefore a basis. $\square$

Problem 12. ► Recitation

Let A be an $m \times n$ matrix and B an $n \times p$ matrix.

- (a) Prove that $\text{Col}(AB) \subseteq \text{Col } A$.
- (b) Prove that $\text{Nul } B \subseteq \text{Nul}(AB)$.
- (c) Give 2×2 examples where $\text{Col}(AB) \neq \text{Col } A$, and where $\text{Nul } B \neq \text{Nul}(AB)$.
- (d) Now let A and B be $n \times n$, with AB invertible. Use (a), (b) and the invertible matrix theorem to prove that A and B are both invertible.

Solution. (a) Let $y \in \text{Col}(AB)$. Then $y = (AB)x$ for some x in $\mathbb{R}^p$. By associativity $y = A(Bx)$, so $y = Az$ with $z = Bx$ in $\mathbb{R}^n$, and $y \in \text{Col } A$. $\square$

(b) Let $x \in \text{Nul } B$, so $Bx = \mathbf{0}$. Then $(AB)x = A(Bx) = A\mathbf{0} = \mathbf{0}$, so $x \in \text{Nul}(AB)$. $\square$

(c) $A = I$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: then $AB = B$, and $\text{Col}(AB) = \text{Span}\{e_1\}$, a line, while $\text{Col } A = \mathbb{R}^2$. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = I$: then $AB = A$, and $\text{Nul}(AB) = \text{Span}\{e_2\}$, while $\text{Nul } B = \{\mathbf{0}\}$.

(d) AB is invertible, so by the invertible matrix theorem its columns span $\mathbb{R}^n$ (statement (8)) and $(AB)x = \mathbf{0}$ has only the trivial solution (statement (3)): $\text{Col}(AB) = \mathbb{R}^n$ and $\text{Nul}(AB) = \{\mathbf{0}\}$. By (a), $\mathbb{R}^n = \text{Col}(AB) \subseteq \text{Col } A \subseteq \mathbb{R}^n$, so $\text{Col } A = \mathbb{R}^n$: the columns of A span $\mathbb{R}^n$, and A is invertible by statement (8). By (b), $\text{Nul } B \subseteq \text{Nul}(AB) = \{\mathbf{0}\}$, so $Bx = \mathbf{0}$ has only the trivial solution, and B is invertible by statement (3). $\square$

(This is a second proof of Proposition 3.6.14, which Lecture 17 proved with one-sided inverses.)

Problem 13. A square matrix is *unit lower triangular* if its entries above the diagonal are 0 and its diagonal entries are 1, like the L of an LU factorisation (Lecture 18).

- (a) Prove that the product of two unit lower triangular $n \times n$ matrices is unit lower triangular.
- (b) Prove that a unit lower triangular matrix L is invertible, and that L^{-1} is unit lower triangular. (Hint: reduce L to I with row replacements, and use elementary matrices.)
- (c) Let A be invertible, with $A = L_1U_1 = L_2U_2$, where L_1, L_2 are unit lower triangular and U_1, U_2 are upper triangular. Prove that $L_1 = L_2$ and $U_1 = U_2$: the LU factorisation of an invertible matrix is unique. (You may use that an invertible upper triangular matrix has an upper triangular inverse, and that a product of upper triangular matrices is upper triangular.)

Solution. (a) Let L and M be unit lower triangular: $l_{ik} = 0$ and $m_{ik} = 0$ when $k > i$, and $l_{ii} = m_{ii} = 1$. Entry (i, j) of LM is $\sum_k l_{ik}m_{kj}$. If $i < j$, every term is 0: either $k > i$, and then $l_{ik} = 0$; or $k \leq i < j$, and

then $m_{kj} = 0$. So LM has zeros above the diagonal. If $i = j$, a term can be nonzero only if $k \leq i$ (for l_{ik}) and $k \geq i$ (for m_{ki}), that is $k = i$; so $(LM)_{ii} = l_{ii}m_{ii} = 1$. $\square$

(b) Gauss–Jordan on L needs no swaps and no scalings: the pivots are the diagonal 1s, and the entries below them are cleared with replacements $R_i \rightarrow R_i - cR_j$ with $j < i$, working column by column from the left. (Each replacement only changes entries to the left of the diagonal in row i , so the diagonal stays 1.) So $E_k \cdots E_1 L = I$, where each E_r is I after one replacement $R_i \rightarrow R_i - cR_j$ with $j < i$: a unit lower triangular matrix with a single nonzero entry $-c$ below the diagonal. Hence L is invertible, $L^{-1} = E_k \cdots E_1$, and by (a), applied repeatedly, L^{-1} is unit lower triangular. $\square$

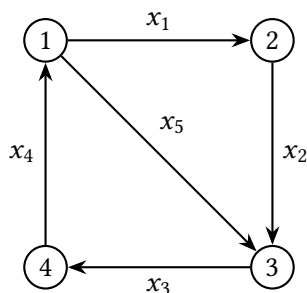
(c) A is invertible, so $U_1 = L_1^{-1}A$ and $U_2 = L_2^{-1}A$ are invertible (products of invertible matrices). From $L_1 U_1 = L_2 U_2$, multiply on the left by L_2^{-1} and on the right by U_1^{-1} :

$$L_2^{-1} L_1 = U_2 U_1^{-1}.$$

The left side is unit lower triangular, by (a) and (b). The right side is upper triangular, by the facts we may use. A matrix that is both has zeros above and below the diagonal, and 1s on it: it is I . So $L_2^{-1} L_1 = I$, which gives $L_1 = L_2$, and $U_2 U_1^{-1} = I$, which gives $U_1 = U_2$. $\square$

D. Challenge and application

Problem 14. A small road network has four junctions and five one-way roads. Traffic is measured in cars per minute: $x_1, \dots, x_5$ on the five roads, as in the figure. At each junction, as many cars arrive as leave.



- Write the four balance conditions as a homogeneous system $Ax = 0$, with one row per junction (cars arriving minus cars leaving).
- Find a basis of $\text{Nul } A$ and its dimension. Describe each basis vector as a loop of traffic in the network.
- The loop $1 \rightarrow 2 \rightarrow 3$ and back from 3 to 1 along road 5, against its direction, is not allowed for real cars, but it is still a vector in $\text{Nul } A$. Write it in terms of your basis.
- Now cars may also enter or leave the network at the junctions: $Ax = b$, where b_i is the number of cars per minute that leave the network at junction i (negative if they enter). Show that every column of A has entries adding up to 0, and deduce that $Ax = b$ can only be solved when $b_1 + b_2 + b_3 + b_4 = 0$. Explain this in words.
- Show that $\dim \text{Col } A = 3$, and conclude that $Ax = b$ has a solution for every b whose entries add up to 0, that is, $b_1 + b_2 + b_3 + b_4 = 0$.

Solution. (a) Junction 1: cars arrive on road 4 and leave on roads 1 and 5, so $x_4 - x_1 - x_5 = 0$. Junction

2: $x_1 - x_2 = 0$. Junction 3: $x_2 + x_5 - x_3 = 0$. Junction 4: $x_3 - x_4 = 0$. So

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

(b) Row reduce. $R_2 \rightarrow R_2 + R_1$, then $R_3 \rightarrow R_3 + R_2$, then $R_4 \rightarrow R_4 + R_3$:

$$A \rightarrow \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Pivots in columns 1, 2, 3; $x_4 = s$ and $x_5 = t$ are free. Back substitution: $x_3 = s$, $x_2 = s - t$, $x_1 = s - t$. So

$$\mathbf{x} = s(1, 1, 1, 1, 0) + t(-1, -1, 0, 0, 1),$$

a basis of $\text{Nul } A$ is $((1, 1, 1, 1, 0), (-1, -1, 0, 0, 1))$, and $\dim \text{Nul } A = 2$. The first is one car per minute around the outer loop $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$. The second sends a car along road 5 from 1 to 3 and takes one away from roads 1 and 2: it is the loop $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$, going around the upper triangle with road 5 forwards and roads 2 and 1 backwards. A basis made of loops that cars can actually drive is $(1, 1, 1, 1, 0)$ and $(0, 0, 1, 1, 1)$ ($1 \rightarrow 3 \rightarrow 4 \rightarrow 1$), the sum of the two vectors above. Both are in $\text{Nul } A$, they are independent (look at entries 1 and 5), and two independent vectors in a 2-dimensional subspace form a basis.

(c) The loop is $(1, 1, 0, 0, -1)$: one unit on roads 1 and 2, and -1 on road 5. It is -1 times the second basis vector: $(1, 1, 0, 0, -1) = 0(1, 1, 1, 1, 0) - 1(-1, -1, 0, 0, 1)$. Check: $A(1, 1, 0, 0, -1) = (-1+1, 1-1, 1-1, 0) = \mathbf{0}$.

(d) Each road leaves one junction and arrives at another, so each column of A has one entry 1 and one entry -1 , and adds up to 0. If $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_5\mathbf{a}_5$, and adding the four entries gives $b_1 + b_2 + b_3 + b_4 = x_1 \cdot 0 + \cdots + x_5 \cdot 0 = 0$. In words: cars are neither created nor destroyed inside the network, so in total as many cars leave the network as enter it.

(e) The echelon form in (b) has three pivots, so the pivot columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ form a basis of $\text{Col } A$ and $\dim \text{Col } A = 3$. By (d), $\text{Col } A$ lies inside $W = \{\mathbf{b} \in \mathbb{R}^4 : b_1 + b_2 + b_3 + b_4 = 0\} = \text{Nul} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, which has one pivot and three free variables, so $\dim W = 3$. A subspace of W with the same dimension as W is all of W (Lecture 20). So $\text{Col } A = W$: every $\mathbf{b}$ with $b_1 + b_2 + b_3 + b_4 = 0$ is some $A\mathbf{x}$.