

Problem Sheet 5

Lectures 17 to 20: the invertible matrix theorem; LU factorisation; subspaces of $\mathbb{R}^n$; basis and dimension in $\mathbb{R}^n$. Problems marked ► are solved in Recitation 5; the others are for practice on your own, with full solutions in the solutions booklet.

A. Warm-up

Problem 1. Recall (Lecture 17): an *elementary matrix* E is what I becomes after one row operation, and EA performs that same row operation on A . Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}.$$

- Which row operation does each of E_1 , E_2 , E_3 perform?
- Compute E_1A , E_2A and E_3A , and check that each one is A after that row operation.
- Write down E_1^{-1} , E_2^{-1} and E_3^{-1} without any calculation, by asking which row operation undoes each one.

Problem 2. Recall (Lecture 19): a *subspace* of $\mathbb{R}^n$ is a subset H that contains $\mathbf{0}$ and is closed under addition and under multiplication by scalars. Which of these sets are subspaces? Prove your answer for each subspace, and show one rule that fails, with numbers, for each set that is not.

- $H_1 = \{(x, y, z) \in \mathbb{R}^3 : x + 3y - 2z = 0\}$
- $H_2 = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$
- $H_3 = \{(x, y) \in \mathbb{R}^2 : y = |x|\}$
- $H_4 = \{(t, 2t, -t) : t \in \mathbb{R}\}$

Problem 3. Recall (Lecture 19): $\text{Nul } A$ is the set of solutions of $A\mathbf{x} = \mathbf{0}$, and $\text{Col } A$ is the span of the columns of A . Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 0 & 3 & -3 \end{bmatrix}.$$

- Is $\mathbf{u} = (2, -2, -2)$ in $\text{Nul } A$? Is $\mathbf{w} = (1, 1, 0)$?
- Is $\mathbf{b} = (1, 5, 3)$ in $\text{Col } A$? Is $\mathbf{c} = (1, 0, 0)$? When the answer is yes, write the vector as a combination of the columns of A .

Problem 4. Recall (Lecture 18): if $A = LU$, then $A\mathbf{x} = \mathbf{b}$ is solved in two triangular steps: first $L\mathbf{y} = \mathbf{b}$ by forward substitution, then $U\mathbf{x} = \mathbf{y}$ by back substitution. Let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}.$$

- Solve $A\mathbf{x} = \mathbf{b}$ for $A = LU$, without computing A .
- Compute $A = LU$, and check your answer to (a).

B. Core

Problem 5. ► Recitation

Use the invertible matrix theorem (Lecture 17, Theorem 3.6.10) to decide which of these matrices are invertible. Do not compute any inverse; name the statement of the theorem that you use.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & -1 \\ 0 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -2 & 0 \\ 4 & 5 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 1 & 1 & -2 \end{bmatrix}.$$

- (a) A , (b) B , (c) C ;
- (d) an $n \times n$ matrix D about which we only know that $D^2 = D + I$;
- (e) all numbers h for which $M_h = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & h \\ 2 & 3 & 5 \end{bmatrix}$ is invertible.

Problem 6. ► Recitation

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 7 \\ -2 & 5 & 1 \end{bmatrix}$.

- (a) Find an LU factorisation of A (Lecture 18): reduce A to an echelon form U using only row replacements, and record the multipliers in L . Check that $LU = A$.
- (b) Use L and U to solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (4, 6, -6)$ and for $\mathbf{b} = (2, 8, 8)$.
- (c) Explain why $B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ has no LU factorisation, and find a permutation matrix P and matrices L, U with $PB = LU$.

Problem 7. ► Recitation

Let $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ -1 & -2 & 1 & -5 \end{bmatrix}$.

- (a) Find the reduced row echelon form R of A .
- (b) Find a basis of $\text{Nul } A$ (Lecture 20), and its dimension.
- (c) Find a basis of $\text{Col } A$, and its dimension. Is $\text{Col } A$ equal to $\text{Col } R$?
- (d) Is $\mathbf{b} = (1, 3, 0)$ in $\text{Col } A$? Is $\mathbf{e}_1 = (1, 0, 0)$?

Problem 8. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$.

- (a) Reduce A to I with three row operations: a swap, a scaling and a replacement. Write down the elementary matrices E_1, E_2, E_3 of these operations, in order, so that $E_3E_2E_1A = I$.
- (b) Deduce A^{-1} as a product of elementary matrices, compute it, and check that $AA^{-1} = I$.
- (c) Write A itself as a product of elementary matrices (Lecture 17), and check the product.

Problem 9. ► Recitation

Let $H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + 2x_2 - x_3 = 0\}$, a plane through the origin.

- Write H as the null space of a matrix, find a basis of H , and give $\dim H$.
- Check that $\mathbf{v} = (1, 1, 3)$ lies in H , and write it in terms of your basis.
- Exchange one vector of your basis for $\mathbf{v}$, so that the new list is still a basis of H . Justify that it is one.
- Extend your basis of (a) to a basis of $\mathbb{R}^3$.
- Is $((-2, 1, 0), (1, 0, 1), (1, 1, 3))$ a basis of $\mathbb{R}^3$?

Problem 10. ► Recitation

Let $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$.

- Find a basis and the dimension of $\text{Nul } A$ and of $\text{Col } A$.
- Is A invertible? Name two statements of the invertible matrix theorem that fail.
- Find all solutions of $A\mathbf{x} = (3, -6)$, and explain why $A\mathbf{x} = (1, 1)$ has none.
- Draw two planes side by side: in the input plane, $\text{Nul } A$ and the solution set of $A\mathbf{x} = (3, -6)$; in the output plane, $\text{Col } A$, the vector $(3, -6)$ and the vector $(1, 1)$.

C. Proof and concept**Problem 11. ► Recitation**

True or false? Prove each true statement and give a counterexample for each false one.

- If A is a 3×3 matrix and the systems $A\mathbf{x} = \mathbf{e}_1$ and $A\mathbf{x} = \mathbf{e}_2$ both have solutions, then A is invertible.
- If H is a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ is a vector not in H , then the set $\mathbf{v} + H = \{\mathbf{v} + \mathbf{h} : \mathbf{h} \in H\}$ is not a subspace.
- Every square matrix that has an LU factorisation is invertible.
- If $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a basis of $\mathbb{R}^3$, then so is $(\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w})$.

Problem 12. ► Recitation

Let A be an $m \times n$ matrix and B an $n \times p$ matrix.

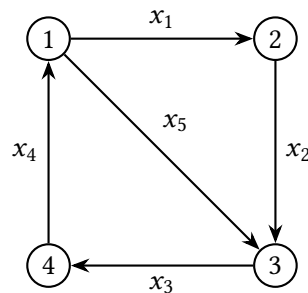
- Prove that $\text{Col}(AB) \subseteq \text{Col } A$.
- Prove that $\text{Nul } B \subseteq \text{Nul}(AB)$.
- Give 2×2 examples where $\text{Col}(AB) \neq \text{Col } A$, and where $\text{Nul } B \neq \text{Nul}(AB)$.
- Now let A and B be $n \times n$, with AB invertible. Use (a), (b) and the invertible matrix theorem to prove that A and B are both invertible.

Problem 13. A square matrix is *unit lower triangular* if its entries above the diagonal are 0 and its diagonal entries are 1, like the L of an LU factorisation (Lecture 18).

- (a) Prove that the product of two unit lower triangular $n \times n$ matrices is unit lower triangular.
- (b) Prove that a unit lower triangular matrix L is invertible, and that L^{-1} is unit lower triangular. (Hint: reduce L to I with row replacements, and use elementary matrices.)
- (c) Let A be invertible, with $A = L_1 U_1 = L_2 U_2$, where L_1, L_2 are unit lower triangular and U_1, U_2 are upper triangular. Prove that $L_1 = L_2$ and $U_1 = U_2$: the LU factorisation of an invertible matrix is unique. (You may use that an invertible upper triangular matrix has an upper triangular inverse, and that a product of upper triangular matrices is upper triangular.)

D. Challenge and application

Problem 14. A small road network has four junctions and five one-way roads. Traffic is measured in cars per minute: $x_1, \dots, x_5$ on the five roads, as in the figure. At each junction, as many cars arrive as leave.



- (a) Write the four balance conditions as a homogeneous system $A\mathbf{x} = \mathbf{0}$, with one row per junction (cars arriving minus cars leaving).
- (b) Find a basis of $\text{Nul } A$ and its dimension. Describe each basis vector as a loop of traffic in the network.
- (c) The loop $1 \rightarrow 2 \rightarrow 3$ and back from 3 to 1 along road 5, against its direction, is not allowed for real cars, but it is still a vector in $\text{Nul } A$. Write it in terms of your basis.
- (d) Now cars may also enter or leave the network at the junctions: $A\mathbf{x} = \mathbf{b}$, where b_i is the number of cars per minute that leave the network at junction i (negative if they enter). Show that every column of A has entries adding up to 0, and deduce that $A\mathbf{x} = \mathbf{b}$ can only be solved when $b_1 + b_2 + b_3 + b_4 = 0$. Explain this in words.
- (e) Show that $\dim \text{Col } A = 3$, and conclude that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ whose entries add up to 0, that is, $b_1 + b_2 + b_3 + b_4 = 0$.